Research statement

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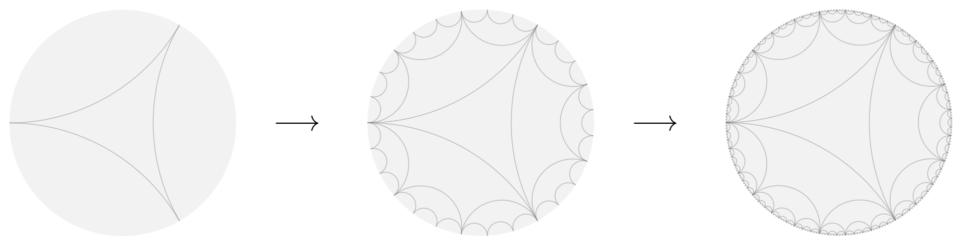
Overview

I am a mathematical physicist interested in statistical-mechanical models that give rise to quantum field theories in their continuum limit. Typically, one begins with finite combinatorial-algebraic objects that define the model in question, together with a map that takes objects on one scale to objects on another scale, for example:



(1)

The iterative application of this 'fine-graining' map gives rise to a continuum theory from the finite statistical-mechanical model. Under many circumstances, desirable properties of the continuum theory can be endowed from the statistical-mechanical model. A main theme of this work is to identify and exploit these circumstances to address open problems in the field.

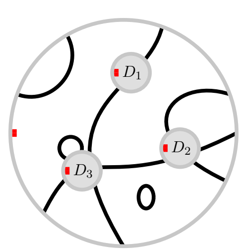
My research activities fit within four well-established research programs:

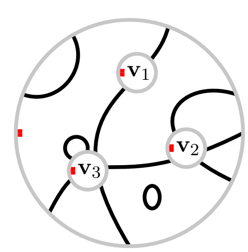
1. Solving statistical-mechanical models equipped with integrable structure [1, 2].
2. The classification of subfactor planar algebras with small dimension [3, 4, 5, 6].
3. Constructing a conformal field theory corresponding to each finite index subfactor [7, 8].
4. Causal dynamical triangulations, a non-perturbative approach to quantum gravity [9, 10, 11].

Following a brief introduction, I elaborate on specific projects related to each of these programs.

Preliminaries: *Planar-algebraic models*

A *planar algebra* is an algebraic object that facilitates the 'planar multiplication' of vectors in a graded vector space $(P_n)_{n\in\mathbb{N}_0}$ [12, 13]. A basis for P_n consists of disks with n *nodes* on their boundary such that when disks are combined ('multiplied'), every node is connected to a single other node via non-intersecting strings. *Planar tangles* are the diagrammatic objects, defined up to ambient isotopy, that facilitate the combination of vectors, for example:

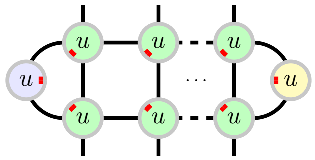
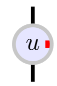
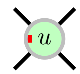
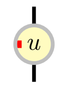
$T =$ 

, $\quad \mathsf{P}_T(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) =$ 

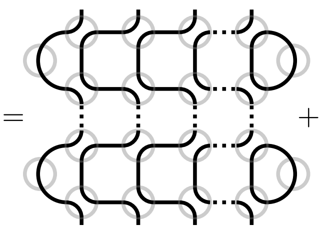
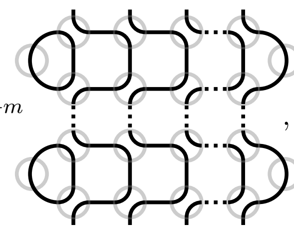
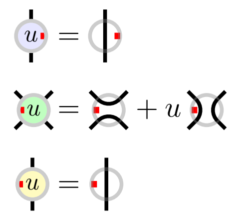
$\in P_8,$ (2)

where $\mathsf{P}_T : P_2 \times P_4 \times P_6 \to P_8$ is the multilinear map corresponding to the tangle T. To contrast, the binary operation $\circ : A \times A \to A$ of an associative algebra A is a 'linear multiplication'. In fact, planar algebras possess linear multiplication tangles that endow each P_{2n} with the structure of an associative algebra. *Subfactor planar algebras* are a particular class of planar algebras that encode the structure of *subfactors* – inclusions of von Neumann algebras with trivial centre, whereby each P_{2n} is isomorphic to a finite-dimensional matrix algebra [12].

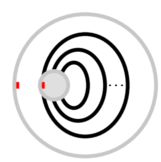
Planar algebras are particularly suited to the description of two-dimensional statistical-mechanical systems as they naturally encode interactions in the plane. In the spirit of Baxter [2] and Sklyanin [14], *planar-algebraic models* are described by a *transfer operator*

$T_n(u) :=$  $\in P_{2n}$,  $\in P_2$,  $\in P_4$,  $\in P_2$, (3)

such that for each $m, n \in \mathbb{N}$ the element $T_n(u)^m$ (which corresponds to m transfer operators stacked on top of one another) generates a $m \times n$ lattice and assigns the appropriate weight to each configuration of the model. A planar algebra *encodes* the algebraic structure of a model if no proper planar subalgebra can take its place. Considering a model with underlying Temperley-Lieb planar-algebraic structure [15], we have

$T_n(u)^m =$  $+ \cdots + u^{n+m}$  ,  . (4)

By construction, the partition function of the model is a function of the transfer operator, the details of which depend on the specific boundary conditions. Taking, for example, periodic boundary conditions and a model with an underlying subfactor planar-algebraic structure, the partition function is given by

$Z_{n,m}(u) := \mathsf{P}_{\mathrm{tr}_n}\big(T_n(u)^m\big)$, $\qquad \mathrm{tr}_n :=$  , (5)

where $\mathsf{P}_{\mathrm{tr}_n}$ is the analogue of the standard $n \times n$ matrix trace. In this case, a *solution* of the model amounts to determining the eigenvalues of the transfer operator, from which the partition function and other useful properties of the model can be determined [2].

Project 1: *Integrable structure of planar-algebraic models*

Background. A planar-algebraic model is *integrable* if the transfer operator satisfies

$$[T_n(u), T_n(v)] = 0, \tag{6}$$

for all $n \in \mathbb{N}$ and for all $u, v \in \Omega$, where $\Omega \subseteq \mathbb{F}$ is a suitable domain. Expanding the transfer operator in a basis of scalar functions, for example, performing a power series in u

$$T_n(u) = \sum_{i=0}^{\infty} u^i Q_i, \qquad \text{integrability implies} \qquad [Q_i, Q_j] = 0, \qquad \forall\, i, j \in \mathbb{N}_0. \tag{7}$$

The commutative P_{2n}-subalgebra generated by the transfer operator is denoted by H_{2n} and is considered the space of *hamiltonians* of the model. The space of *integrals of motion* (IOM) of the model, denoted by I_{2n}, is the subalgebra spanned by elements in the algebra P_{2n} that commute with each element in H_{2n}, and whose elements all mutually commute. Accordingly, we have the following sequence

$$H_{2n} \subseteq I_{2n} \subseteq P_{2n}. \tag{8}$$

Analysing the relative dimensions of algebras in this sequence provides insight into the underlying integrable structure of the model, leading naturally to the following.

Question 1. *What are the typical dimensions of H_{2n} and I_{2n}?*

Question 2. *Which models exhaust the integrable structure endowed from the underlying planar algebra, i.e $H_{2n} = I_{2n}$?*

Progress. A model with the simplest integrable structure has $\dim H_{2n} = 1$. In this case, the transfer operator can be expressed as a polynomial in a single algebraic element, and we refer to the underlying model as *polynomially integrable*. Together with Rasmussen in [16], we present a criterion for polynomially integrable models when P_{2n} is semisimple. We find that models for which $\dim H_{2n} = 1$, are surprisingly ubiquitous, and present the following special case to illustrate.

Proposition. *Let $T_n(u)$ denote a transfer operator corresponding to an integrable model where P_{2n} is semisimple, if $T_n(u)$ is diagonalisable then* $\dim H_{2n} = 1$.

For subfactor planar algebras, where P_{2n} are semisimple for all $n \in \mathbb{N}_0$, the diagonalisability of the transfer operator follows from mild conditions on the underlying green-, blue- and yellow-operators in (3) which, in many instances, are implied from integrability.

For the case $\dim H_{2n} = 1$, models addressed by Question 2 satisfy a rather simple criterion. Let $h \in P_{2n}$ denote the generator of H_{2n}, we have $H_{2n} = I_{2n}$ if and only if h generates its own centraliser in P_{2n}. Also in [16], we find that a model with underlying Temperley-Lieb planar-algebraic structure satisfies this criterion for $n = 1, \ldots, 17$.

Future work.

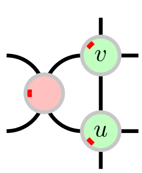
- Construct models with $\dim H_{2n} > 1$. A consequence of the criterion in [16], is that these models necessarily possess a non-diagonalisable transfer operator. Models with an underlying Temperley-Lieb algebraic structure, for particular values of the loop fugacity, are known to admit this behaviour [17] and therefore are natural candidates.

- Establish maps $H_{2n} \to H_{2(n+1)}$ and $I_{2n} \to I_{2(n+1)}$ that facilitate analysis of continuum limits of hamiltonians and IOM.

- While the planar-algebraic models introduced here are defined on the strip, it is straightforward to extend this construction and define transfer operators on the cylinder. In this case much of the details carry over. One key difference is that the algebraic structure is endowed from the affine category of a given planar algebra which, in some basic cases, is infinite-dimensional. It would be interesting to generalise the criterion such that it implies $\dim H_{2n} = 1$ within an infinite-dimensional setting.

Project 2: *Classification of Yang-Baxter integrable models*

Background. A planar-algebraic model is *Yang-Baxter integrable* if (6) is a consequence of a set of local relations satisfied by the blue-, green- and yellow-operators in (3), which include the celebrated Yang-Baxter equation (YBE) [18, 19, 1]



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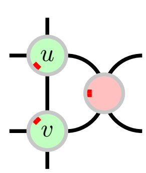

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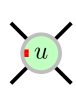
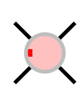
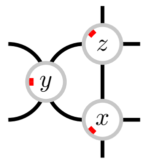


$= \sum_{a \in B_4} r_a(u)\, a,$

$= \sum_{a \in B_4} y_a(u)\, a,$ (9)

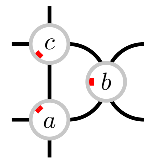
where B_4 is a basis for P_4, and we highlight that the red-operator is considered 'auxiliary' and is parameterised independently from the green-operators. We are led naturally to the following.

Question 3. *Which planar algebras give rise to Yang-Baxter integrable models?*

Let us first consider planar algebras that *do not* give rise to Yang-Baxter integrable models. At the level of the planar algebra the YBE imposes a set of *Yang-Baxter relations* (YBRs)



$$= \sum_{a,b,c\in B_4} C^{a,b,c}_{x,y,z}$$



, (10)

for some $C^{a,b,c}_{x,y,z} \in \mathbb{F}$. Planar algebras that *do not* satisfy the appropriate set of YBRs *do not* admit a Yang-Baxter integrable model, while the converse need not be true. A planar algebra will be called *YBR consistent* if it satisfies the appropriate set of YBRs. Accordingly, we present a refined version of Question 3: which YBR consistent planar algebras admit Yang-Baxter integrable models? As a first step towards addressing this question, we consider the special case of *YBR planar algebras.*

A YBR planar algebra is a subfactor planar algebra whereby each triple of vector in $x, y, z \in P_4$ satisfy a YBR [6]. By construction, each YBR planar algebra is YBR consistent, and is therefore a possible class of planar algebras addressed by Question 3.

Question 4. *Do all YBR planar algebras give rise to Yang-Baxter integrable models?*

Progress. The simplest YBR planar algebra is the Temperley-Lieb subfactor planar algebra, which has long been known to admit a Yang-Baxter integrable model [20, 21] (see also [22]). In fact, this is the model presented in (4)! The next step up in complexity are the so-called *singly generated YBR planar algebras* [6], whose basis B_4 consists of the two canonical Temperley-Lieb basis vectors and *one* additional vector, hence the terminology. Conveniently, singly generated YBR planar algebras admit the following classification due to Liu [6].

Theorem. *A singly-generated YBR planar algebra is isomorphic to a quotient of a Fuss-Catalan (FC), Birman-Wenzl-Murakami (BMW) or Liu planar algebra.*

The FC [23] and BMW [24, 25] planar algebras are both known to underlie Yang-Baxter integrable models [26, 27], while until recently no such model was known to exist for the Liu planar algebra. In forthcoming work with Rasmussen [28], we make use of the algebraic integrability framework [16] to construct a Yang-Baxter integrable model with an underlying Liu planar algebraic structure, and in so doing, provide a positive answer to Question 4 for the case of singly generated YBR planar algebras.

Future work.

- Extend Liu's classification to doubly-generated (and possibly higher) planar algebras and establish whether or not the resulting algebras underlie Yang-Baxter integrable models.

- For YBR planar algebras, YBRs are imposed for every triple of vectors in the basis. Under some circumstances this is unnecessary, for example, when the green- and red-operators in (9) have basis vectors for which the corresponding coefficients are zero. It is therefore likely that the answer to the converse of Question 4, is no. In light of these expectations, it would be instructive to find explicit examples of Yang-Baxter integrable models that *do not* have an underlying YBR planar-algebraic structure.

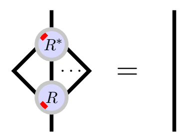
- Informed by the presence or absence of these examples, find a class of planar algebras that incorporate such models and, in turn, develop a conjectured answer to Question 3.

Project 3: *Conformal field theories from planar algebras*

Background. A conformal net is an axiomatisation of a chiral conformal field theory on the circle [29]. For our purposes, we highlight two key features (i) each closed interval $I \subset S^1$ is assigned an algebra $\mathcal{A}(I)$, acting on a Hilbert space $\mathcal{H}$ (ii) the action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$, induces a continuous unitary representation. Remarkably, conformal nets and subfactors share many common features. In fact, it is possible to associate a subfactor to each conformal net [30]. However, the status of the converse remains unclear. To clarify this point, Jones posed the following [7].

Question 5. *Does each subfactor arise from quantum field theory?*

In an attempt to answer this question, Jones introduced a machinery whereby planar algebras provide 'semicontinuous' models of conformal nets [7, 8]. In these models, nodes on the exterior of planar algebra disks approximate S^1, which are restricted to correspond to *k-adic partitions of* S^1 for $k \in \mathbb{N}_{>1}$. Similarly, each $I \subset S^1$, are restricted to be consistent with a k-adic partition of S^1, and index an algebra of operators within a planar algebra, which take the role of $\mathcal{A}(I)$. The action of these algebras on the semicontinuous Hilbert space – the analogue of $\mathcal{H}$, together with an element $R \in P_{k+1}$ satisfying the isometry condition:



(11)

induce a unitary representation of Thompson groups T_k [7]. Elements of T_k act on k-adic partitions of S^1 via scaling and rotations [31], and approximate elements of $\mathrm{Diff}_+(S^1)$ to arbitrary precision [32].

If one could take a limit of the semicontinuous model such that the representation of T_k tends toward a continuous representation of $\mathrm{Diff}_+(S^1)$, one recovers a fully fledged conformal net. Jones himself called into question the validity of this approach by presenting a 'no-go theorem' for a particular planar algebra (with unique isometry R) where it was shown, for T_2, that representations of rotations by dyadic rationals are discontinuous [8]. This work was later supplemented by Kliesch and König who showed, for a generic choice of the isometry R within a tensor planar algebra, that the corresponding representation of T_2 is almost surely discontinuous [33].

We highlight that both of these results specialise $k = 2$. To associate a subfactor to each conformal net one simply has to find *a single* $k \in \mathbb{N}_{>1}$ and *a single* $R \in P_{k+1}$ such that the corresponding representation of T_k is continuous. For subfactor planar algebras the dimension of P_{k+1} grows at least exponentially with k, in which case the isometry condition does little to constrain the element R as k is increased. Given the number of possible elements R, one may consider whether some give rise to a continuous representation of T_k, motivating the following.

Question 6. *Are there conditions on $R \in P_{k+1}$ that imply that the corresponding representation of T_k is continuous?*

Progress. In forthcoming work [**?**], I deduce sufficient conditions that imply Jones' machinery induces a continuous representation of the rotation subgroup of T_k, and have found multiple solutions for the Brauer planar algebra. The sufficient conditions have the flavour of YBEs, that is, they are *local* conditions that imply a *global* property of the model. Moreover, these conditions directly avert the type of arguments made in the no-go theorem [8]. Having established continuity of representations of the rotation subgroup, showing continuity for all of T_k reduces to establishing continuity of the subgroup F_k, whose elements act on k-adic partitions of $[0, 1]$ via scaling [31].

Future work.

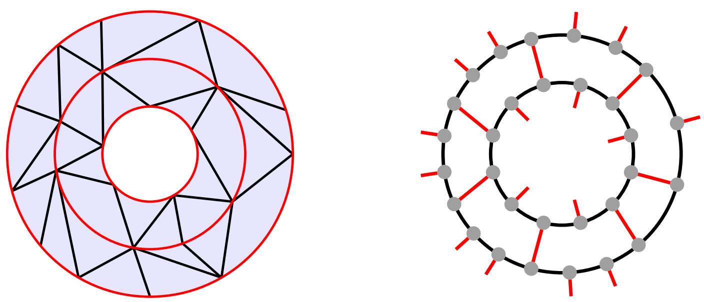
- Derive new solutions to the continuity sufficient conditions with different underlying planar algebraic structure.

- Develop arguments and/or sufficient conditions to establish the continuity of representations of the subgroup F_k.
- Combine arguments/sufficient conditions for continuity of representations of F_k with those that are known for the rotation subgroup to produce continuous representations of T_k.
- Develop the limit that takes continuous representations of T_k to continuous representations of $\mathrm{Diff}_+(S^1)$.
- Inspired by YBR planar algebras which are endowed the algebraic structure of the YBE, we seek to define planar algebras that are endowed the algebraic structure of the continuity sufficient conditions.

Project 4: $1+1$-*dimensional causal dynamical triangulations*

Background. Causal dynamical triangulations (CDT) is a non-perturbative approach to quantum gravity, defined on a triangulated lattice, whose continuum limit is hoped to give rise to a predictive physical theory [9, 10, 11]. Specialising to one dimension of space and one dimension of time, the corresponding models are naturally described by two-dimensional statistical-mechanical systems.

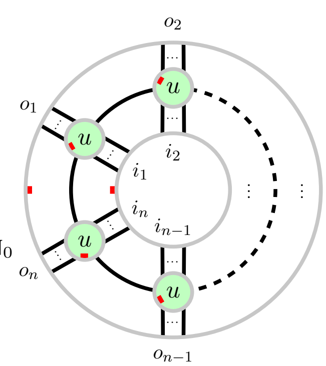
A *two-dimensional causal triangulation of the annulus* is defined by a sequence of circular graphs $S_1, \ldots, S_m$, where $m \in \mathbb{N}$ is the height, such that the annulus between two cycles is triangulated. For example:



(12)

where on the left we present a triangulation and on the right the corresponding dual triangulation. From the perspective of the planar-algebraic framework, the dual of a causal triangulation is highly suggestive. The concentric circles of dual nodes indicate a natural multiplicative structure, while the dual nodes themselves can be elevated to input disks – indicating a generality beyond pure triangulations. Accordingly, we define the transfer operator

$$T(u) := \sum_{n\in\mathbb{N}} \sum_{\substack{\mathbf{i}\in\mathbb{N}^n \\ \mathbf{o}\in\mathbb{N}^{n-1}\times\mathbb{N}_0}}$$



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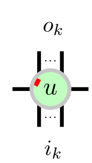

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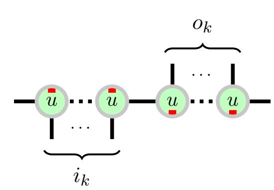

, (13)

where $\mathbf{i} := (i_1, \ldots, i_n)$ and $\mathbf{o} := (o_1, \ldots, o_n)$, such that the element $T(u)^m$ (which corresponds to m transfer operators stacked radially) generates all $(m+1)$-height causal triangulations of the annulus and assigns the appropriate weight to each configuration of the model. By specialising the underlying planar algebra and the corresponding elementary operators of P_3 (appearing in the right-most equation in (13)), one defines a particular CDT model. For example, specialising to the tensor planar algebra and fixing

(14)

one defines a pure gravity model. Considering more elaborate parameterisations is akin to introducing a matter coupling to the purely geometric model (14).

Question 7. *Which couplings give rise to a non-trivial interaction between matter and geometry?*

Progress. In [34] together with collaborators, we define a dense loop model and a dilute loop model on causal triangulations – both with an underlying tensor planar algebra structure. For reference, the elementary operators of the dense model are defined

(15)

We show that the dense loop model exhibits the same critical behaviour and large-scale structure as the pure gravity model (14). In contrast, the dilute loop model is shown to exhibit a difference in both critical behaviour and large-scale structure manifesting in a shift in the Hausdorff dimension of the model. Accordingly, the dilute loop model experiences an influential interaction between matter and geometry – providing an example of a model addressed by Question 7.

Future work.

- Continue to develop models with a non-trivial interaction between matter and geometry.
- Incorporating topology change into two-dimensional CDT has proven a relevant feature when recovering known results [9, 35]. Moreover, planar-algebraic models are particularly amenable to the modelling of various two-dimensional topologies. We seek to define a transfer operator that includes contributions from configurations that experience topology change.
- There have been few examples of integrable structure within CDT, for instance [36]. We seek develop sufficient conditions, inspired by the Yang-Baxter equation on the square lattice, that imply models on causal triangulations are integrable.

Summary of research results

Project 1: *Integrable structure of planar-algebraic models*

- Developed a framework that assigns a statistical mechanical model to each planar algebra and determined sufficient conditions for the model to be integrable.
- Introduced the notion of *polynomial integrability*, and provided sufficient and necessary conditions for models to have this property.
- Showed that for the Temperley-Lieb planar algebra, the corresponding model is polynomially integrable for all $n = 1, \ldots, 17$. Also demonstrated that an 8-vertex model within the tensor planar algebra is polynomially integrable for all relevant parameter values.

Output: [16]

Project 2: *Classification of Yang-Baxter integrable models*

- Showed that a singly-generated planar algebra gives rise to an integrable model (of the form developed in the previous project) if and only if the algebra is a YBR planar algebra.
- Demonstrated that these integrable models are also polynomially integrable.
- A crucial aspect of this classification was the creation of an integrable model for the Liu planar algebra. To our knowledge, this is a completely new model, and the construction required the generality of the integrability framework of the previous project.

Output: [28]

Project 3: *Conformal field theories from planar algebras*

- Identified a set of sufficient conditions that endow representations of the rotation subgroup of Thompson's group T_k with the property of continuity.
- For the Brauer planar algebra, determined an infinite set of solutions endowing continuity to representations of the rotation subgroup of T_{2k+5} for all $k \in \mathbb{N}$.
- Developed a notion of a *discrete conformal net.*

Project 4: $1+1$-*dimensional causal dynamical triangulations*

- Proposed two new models of matter fields coupled to causal triangulations, a dense loop model and a dilute loop model.
- Demonstrated that the dense loop model has the same critical behaviour and large-scale structure as a purely gravitational model.
- In contrast, the dilute loop model was observed to have a different critical behaviour and large-scale structure, resulting in a shift in the Hausdorff dimension of the model.

Output: [34]

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