Towards continuous representations of Thompson groups T_k

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Conformal nets

Question (Jones 2017)

Does each subfactor (planar algebra) give rise to a conformal field theory?

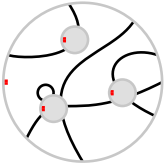
A **conformal net** consists of (i) a Hilbert space $\mathcal{H}$, (ii) a *von Neumann algebra* $\mathcal{A}(I)$ on $\mathcal{H}$ for each open interval $I \subset S^1$, (iii) a continuous unitary representation U of $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$. Subject to:

Isotony:	$\mathcal{A}(I) \subseteq \mathcal{A}(J)$	if $I \subseteq J$
Locality:	$[\mathcal{A}(I), \mathcal{A}(J)] = 0$	if $I \cap J = \{\}$
Covariance:	$U(\alpha)\mathcal{A}(I)U(\alpha)^* = \mathcal{A}(\alpha(I))$	$\alpha \in \mathrm{Diff}_+(S^1)$
Positivity:	$\mathrm{Spec}(U(\rho)) \subset \mathbb{R}^+$	$\rho \in \mathrm{Rot}(S^1)$

Planar algebras

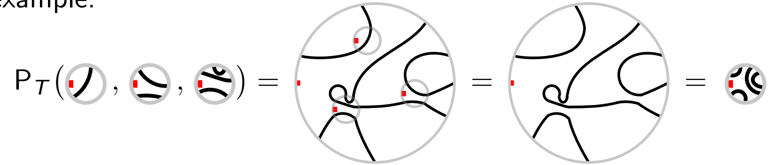
Definition

An (unshaded) **planar algebra** P is a collection of vector spaces $(P_n)_{n\in\mathbb{N}_0}$, together with the action of planar tangles as multilinear maps e.g.

$T =$  , $\mathrm{P}_T : P_2 \times P_4 \times P_6 \to P_8$

such that this action is compatible with the composition of tangles.

For example:



Subfactor planar algebras have an inner product on each $(P_n)_{n\in\mathbb{N}_0}$.

Semicontinuous models

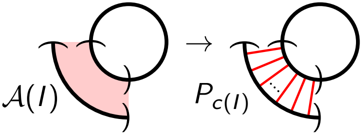
Semicontinuous models are lattice regularisations of conformal nets

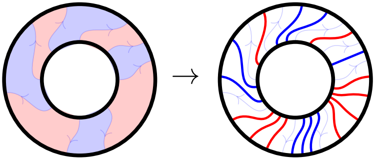
$(\mathcal{H}, \mathcal{A}(I)) \to (P, P)$

Planar algebra

$\mathrm{Diff}_+(S^1) \to T_k$

Thompson group





No-go?

The idea: semicontinuous models $\rightarrow$ conformal nets

The issue: Covariance: $U(\alpha)\mathcal{A}(I)U(\alpha)^* = \mathcal{A}(\alpha(I)) \quad \alpha \in \mathrm{Diff}_+(S^1)$

Reps. of T_k are projective and unitary, but not continuous!

The dream: Develop sufficient conditions that endow reps. of T_k with the property of continuity

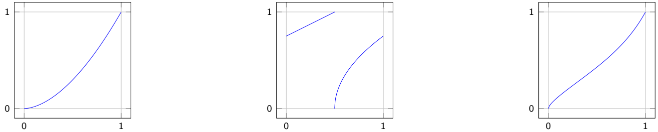
Outline

1. Thompson groups T_k
2. Representations of T_k
3. Continuity conditions
4. Outlook

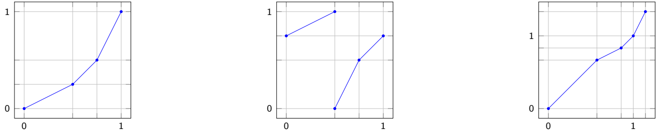
Thompson groups T_k

$\mathrm{Diff}_+(S^1)$ and T_k

Elements of $\mathrm{Diff}_+(S^1)$ can be conveniently expressed as functions:



Elements of T_k can be viewed as ‘discretisations’ of $\mathrm{Diff}_+(S^1)$ elements:



k-adic rational $\left(\frac{a}{k^n}\right)$ break-points and gradients given by powers of k.

Theorem (Zhuang 2007)

For every $f \in \mathrm{Diff}_+(S^1)$ there exists $g \in T_k$, and $\epsilon > 0$ such that

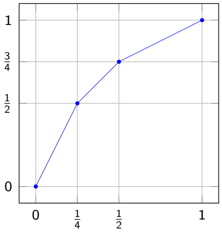
$$\sup_{x \in S^1} |g(x) - f(x)| < \epsilon.$$

Tree diagrams

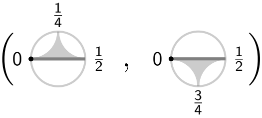
Proposition (Brown 1987)

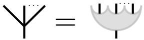
Elements of T_k can be expressed as pairs of annular k-trees.

For $k = 2$ we present the example:



Corresponds to:





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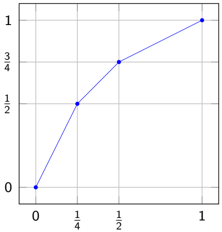

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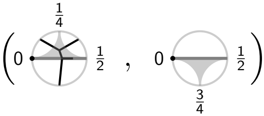
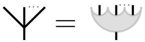


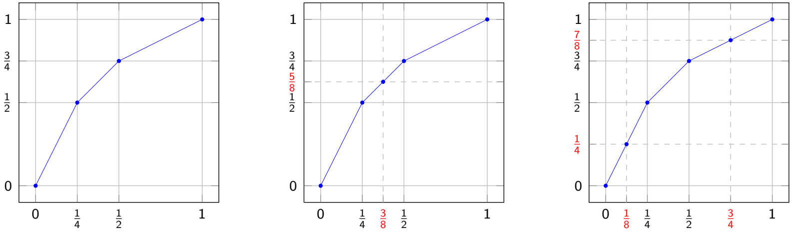


Tree diagrams

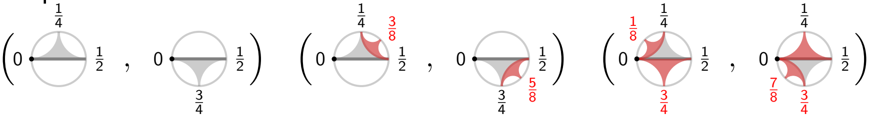
Proposition (Brown 1987)

Elements of T_k can be expressed as pairs of annular k-trees.

Many pairs of annular k-trees give rise to the same element of T_k:



Correspond to:



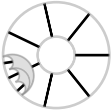
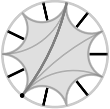
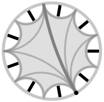
where the highlighted branches correspond to unnecessary divisions.

Forest categories

Denote by $A\mathfrak{F}_k$ the category of annular k-forests:

- $\mathrm{Obj}_{A\mathfrak{F}_k} = \mathbb{N}$
- $\mathrm{Mor}_{A\mathfrak{F}_k}(m,n)$ are annular k-forests with m roots and n leaves

Composition:

$p =$ , $a =$ , $p \circ a =$  $=$ 

where $p \in \mathrm{Mor}_{A\mathfrak{F}_2}(7,9)$ and $a \in \mathrm{Mor}_{A\mathfrak{F}_2}(1,7)$. Define

$$\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathrm{Mor}_{A\mathfrak{F}_k}(1,n)$$

as the set of all annular k-trees, and denote $\ell(f) := \mathrm{target}(f)$ for $f \in \mathcal{D}$.

Fraction notation

Define $\sim$ on pairs of annular k-trees as $(a, y) \sim (b, z)$ if and only if there exist $r, s \in \mathrm{Mor}_{A\mathfrak{F}_k}$ such that $(r \circ a, r \circ y) = (s \circ b, s \circ z)$

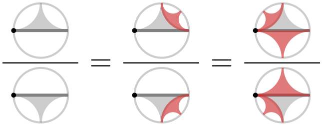
Proposition (Brown 1987)

Two pairs of annular k-trees (a, y) and (b, z) correspond to the same element in T_k if and only if $(a, y) \sim (b, z)$.

Denote by $[(c, x)] \equiv \frac{c}{x}$ the equivalence class (c, x)

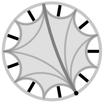
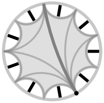
$$\frac{p \circ c}{p \circ x} = \frac{c}{x},$$

where we interpret $p \in \mathrm{Mor}_{A\mathfrak{F}_k}$ as being 'cancelled' in the fraction. Taking the trees from a previous slide:



Composition via fractions

Any $a, b \in \mathcal{D}$ admit $p, q \in \mathrm{Mor}_{A\mathfrak{F}_k}$ such that $p \circ a = q \circ b$, called a *stabilisation*:

$p \circ a =$  $=$  $q \circ b =$  $=$ 

Composition of functions can be expressed as the product of fractions

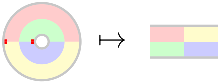
$$G \circ H = \frac{a}{b}\,\frac{c}{d} = \frac{\cancel{q \circ a}}{q \circ b}\,\frac{p \circ c}{\cancel{p \circ d}} = \frac{p \circ c}{q \circ b}, \qquad \text{where } q \circ a = p \circ d,$$

where $G \circ H$ has the domain of H and the range of G.

Representations of T_k

Preliminaries

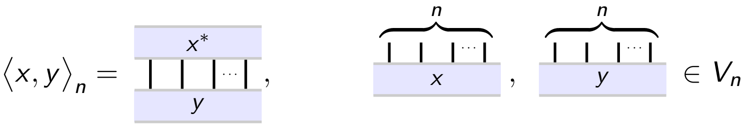
For convenience we will use the cutting convention:



Denote by Hilb the category of Hilbert spaces:

- $\mathrm{Obj}_{\mathsf{Hilb}}$ are Hilbert spaces V_n for each $n \in \mathbb{N}$
- $\mathrm{Mor}_{\mathsf{Hilb}}(V_m, V_n)$ are linear maps

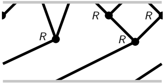
where the inner product on each V_n can be expressed diagrammatically as:

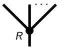


Jones' action

Define the functor $\Phi : A\mathfrak{F}_k \to \text{Hilb}$

- $\Phi_0(n) = V_n$ for all $n \in \mathbb{N}$
- $\Phi_1^R(p) \in \mathrm{Mor}_{\text{Hilb}}(V_m, V_n)$ for all $p \in \mathrm{Mor}_{A\mathfrak{F}_k}(m, n)$

$p =$  , $\Phi_1^R(p) =$ 

, 

$\in P_{k+1}$

Construct the set A_Φ such that:

	T_k	A_Φ
Elements	$\frac{f}{g} = \frac{p \circ f}{p \circ g}$	$\frac{f}{x} = \frac{p \circ f}{\Phi_1^R(p)(x)}$
Action of T_k	$\frac{f_1}{g_1} \frac{f_2}{g_2} = \frac{q \circ f_2}{p \circ g_1}$	$\frac{f_1}{x_1} \frac{f_2}{g_2} = \frac{\cancel{p \circ f_1}}{\Phi_1^R(p)(x_1)} \frac{q \circ f_2}{\cancel{q \circ g_2}} = \frac{q \circ f_2}{\Phi_1^R(p)(x_1)}$

where $x \in V_{\ell(f)}$, $x_1 \in V_{\ell(f_1)}$. Inducing Hilbert space $\mathfrak{H}$ and a representation

$$\pi_R : T_k \to \mathrm{End}(A_\Phi).$$

Continuity conditions

Continuous representations

Definition

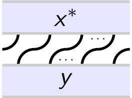
A representation π is *continuous* if each sequence $(f_n)_{n\in\mathbb{N}} \subset T_k$ satisfies

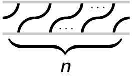
$$\lim_{n\to\infty} \|f_n - \mathrm{id}\| = 0, \qquad \lim_{n\to\infty} \langle \xi, \pi(f_n)(\eta)\rangle = \langle \xi, \eta\rangle, \qquad \forall \xi, \eta \in \mathfrak{H}.$$

Denote by Rot_k the rotation subgroup of T_k, generated by:

$$\varrho_s : S^1 \to S^1, \qquad\qquad x \mapsto x + s \mod 1,$$

where s is a k-adic rational. Matrix elements can be expressed as:

$$\Big\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{k^r}})(\frac{i}{y})\Big\rangle =$$ 

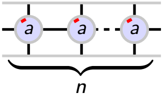
$$= \big\langle x, \Omega_{k^r} y\big\rangle_{k^r}, \quad \Omega_n :=$$ 

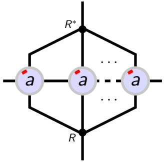
where $x, y \in V_{k^r}$.

The matrix elements can be expressed in terms of a *transfer operator*:

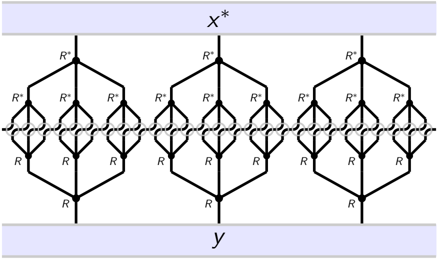
$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{k^{r+s}}})(\frac{i}{y}) \right\rangle = \left\langle x, T_{k^r}(\mathfrak{R}^s(v))y \right\rangle_{k^r}, \qquad v :=$$ 

where $x, y \in V_{k^r}$, and

$T_n(a) =$ 

$\mathfrak{R}(a) =$  .

To illustrate, we present a small example where $(k, r, s) = (3, 1, 2)$:

$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{3^3}})(\frac{i}{y}) \right\rangle =$$ 

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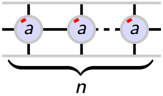
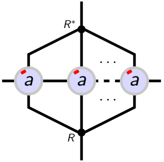
$$T_n(a) =$$ 

$$\mathfrak{R}(a) =$$  .

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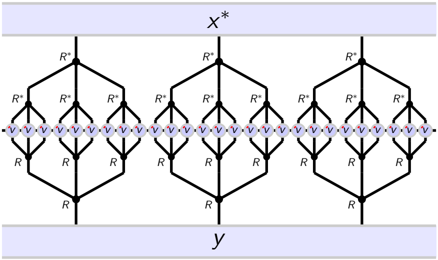
$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{3^3}})(\frac{i}{y}) \right\rangle =$$ 

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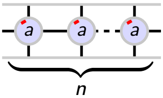

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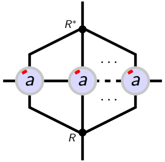

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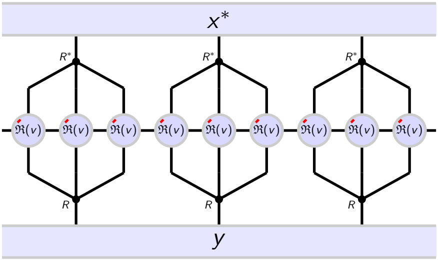

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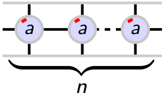

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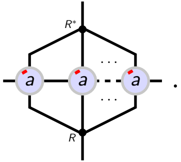

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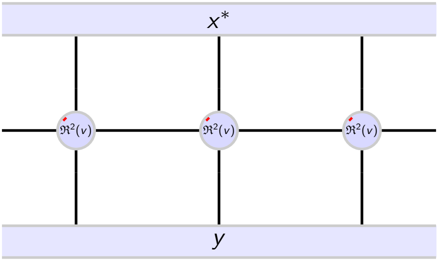

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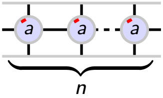

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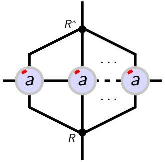

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$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{3^3}})(\frac{i}{y}) \right\rangle = \left\langle x, T_3(\mathfrak{R}^2(v))y \right\rangle_3$$

Continuity conditions

Applying:

$$\lim_{s\to\infty}\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{k^{r+s}}})(\frac{i}{y})\right\rangle = \lim_{s\to\infty}\left\langle x, T_{k^r}(\mathfrak{R}^s(v))y\right\rangle_{k^r} \overset{(!)}{=} \left\langle x, y\right\rangle_{k^r}$$

Proposition

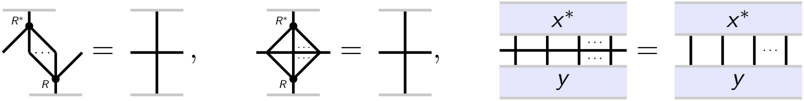
If there exists a $w \in P_4$ such that

$$\mathfrak{R}(v) = w, \qquad \mathfrak{R}(w) = w, \qquad \left\langle x, T_{k^r}(w)y\right\rangle_{k^r} = \left\langle x, y\right\rangle_{k^r} \qquad (\star)$$

for all $x, y \in V_{k^r}$, then

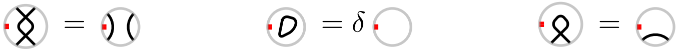
$$\lim_{s\to\infty}\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{k^{r+s}}})(\frac{i}{y})\right\rangle = \left\langle x, y\right\rangle_{k^r}.$$

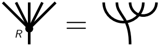
A concrete realisation of $(\star)$ is given by:



Brauer algebra solution

The Brauer planar algebra $(B_n)_{n\in 2\mathbb{N}_0}$ is generated by the action of planar tangles on the space $B_4 = \mathrm{span}(\{$  $\})$, subject to:



Specialising $k = 5$ and $\delta = 1$ we have the solution 

This solution can be generalised to $k = 2n + 5$ for all $n \in \mathbb{N}_0$

Theorem

For $R \in P_{2n+5}$ above, π_R is a continuous unitary representation of Rot_{2n+5}.

Brauer algebra solution

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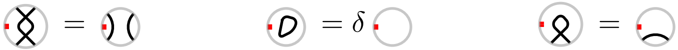
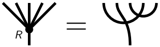


Specialising $k = 5$ and $\delta = 1$ we have the solution 

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Outlook

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Summary:

- Semicontinuous models of conformal nets via planar algebras.
- Limited by the continuity of the representations of T_k.
- Developed sufficient conditions that imply continuity of representations of the rotation subgroup of T_k.

Future work:

- Solve the continuity conditions for other types of planar algebras.
- Construct sufficient conditions that implies the continuity of representation of all T_k
- Develop the limit that takes continuous representations of T_k to continuous representations of $\mathrm{Diff}_+(S^1)$.

The end!