

Towards continuous representations of Thompson groups T_k

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Conformal nets

Question (Jones 2017)

Does each subfactor (planar algebra) give rise to a conformal field theory?

A **conformal net** consists of (i) a Hilbert space \mathcal{H} , (ii) a *von Neumann algebra* $\mathcal{A}(I)$ on \mathcal{H} for each open interval $I \subset S^1$, (iii) a continuous unitary representation U of $\text{Diff}_+(S^1)$ on \mathcal{H} . Subject to:

$$\text{Isotony: } \mathcal{A}(I) \subseteq \mathcal{A}(J) \quad \text{if } I \subseteq J$$

$$\text{Locality: } [\mathcal{A}(I), \mathcal{A}(J)] = 0 \quad \text{if } I \cap J = \emptyset$$

$$\text{Covariance: } U(\alpha)\mathcal{A}(I)U(\alpha)^* = \mathcal{A}(\alpha(I)) \quad \alpha \in \text{Diff}_+(S^1)$$

$$\text{Positivity: } \text{Spec}(U(\rho)) \subset \mathbb{R}^+ \quad \rho \in \text{Rot}(S^1)$$

Planar algebras

Definition

An (unshaded) **planar algebra** P is a collection of vector spaces $(P_n)_{n \in \mathbb{N}_0}$, together with the action of planar tangles as multilinear maps e.g.

$$T = \text{[Diagram of a tangle with 8 strands and 3 shaded regions]}, \quad P_T : P_2 \times P_4 \times P_6 \rightarrow P_8$$

such that this action is compatible with the composition of tangles.

For example:

$$P_T(\text{[Diagram 1]}, \text{[Diagram 2]}, \text{[Diagram 3]}) = \text{[Diagram 4]} = \text{[Diagram 5]} = \text{[Diagram 6]}$$

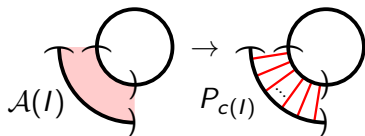
Subfactor planar algebras have an inner product on each $(P_n)_{n \in \mathbb{N}_0}$.

Semicontinuous models

Semicontinuous models are lattice regularisations of conformal nets

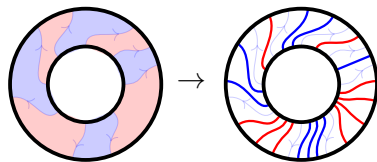
$$(\mathcal{H}, \mathcal{A}(I)) \rightarrow (P, P)$$

Planar algebra



$$\text{Diff}_+(S^1) \rightarrow T_k$$

Thompson group



No-go?

The idea: semicontinuous models \rightarrow conformal nets

The issue: **Covariance:** $U(\alpha)\mathcal{A}(I)U(\alpha)^* = \mathcal{A}(\alpha(I)) \quad \alpha \in \text{Diff}_+(S^1)$
Reps. of T_k are projective and unitary, but not continuous!

The dream: Develop sufficient conditions that endow reps. of T_k with the property of continuity

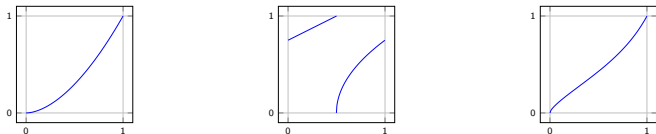
Outline

- 1 Thompson groups T_k
- 2 Representations of T_k
- 3 Continuity conditions
- 4 Outlook

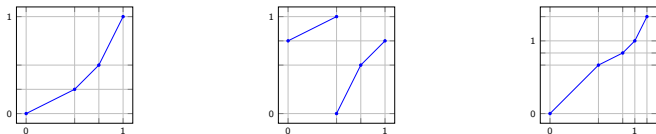
Thompson groups T_k

$\text{Diff}_+(S^1)$ and T_k

Elements of $\text{Diff}_+(S^1)$ can be conveniently expressed as functions:



Elements of T_k can be viewed as 'discretisations' of $\text{Diff}_+(S^1)$ elements:



k -adic rational ($\frac{a}{k^n}$) break-points and gradients given by powers of k .

Theorem (Zhuang 2007)

For every $f \in \text{Diff}_+(S^1)$ there exists $g \in T_k$, and $\epsilon > 0$ such that

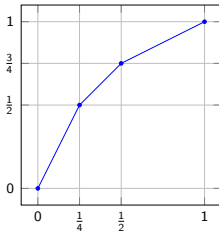
$$\sup_{x \in S^1} |g(x) - f(x)| < \epsilon.$$

Tree diagrams

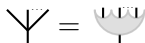
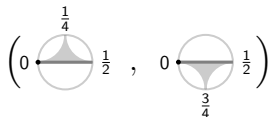
Proposition (Brown 1987)

Elements of T_k can be expressed as pairs of annular k -trees.

For $k = 2$ we present the example:



Corresponds to:

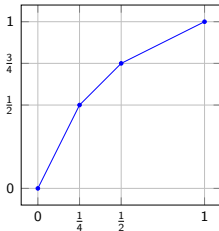


Tree diagrams

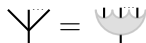
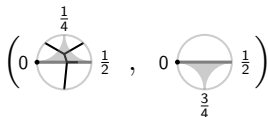
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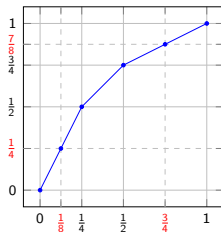
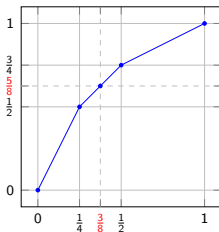
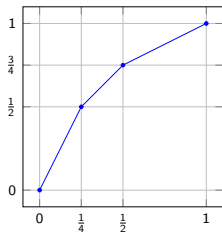


Tree diagrams

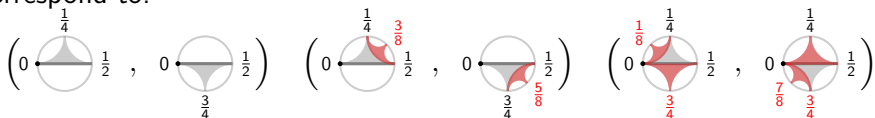
Proposition (Brown 1987)

Elements of T_k can be expressed as pairs of annular k -trees.

Many pairs of annular k -trees give rise to the same element of T_k :



Correspond to:



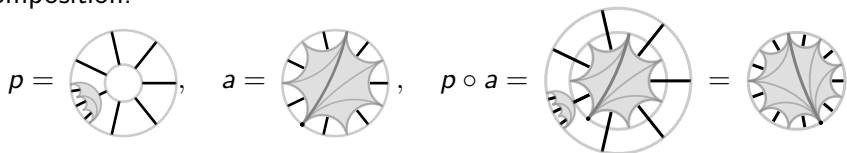
where the highlighted branches correspond to unnecessary divisions.

Forest categories

Denote by $A_{\mathfrak{F}_k}$ the category of annular k -forests:

- $\text{Obj}_{A_{\mathfrak{F}_k}} = \mathbb{N}$
- $\text{Mor}_{A_{\mathfrak{F}_k}}(m, n)$ are annular k -forests with m roots and n leaves

Composition:



where $p \in \text{Mor}_{A_{\mathfrak{F}_2}}(7, 9)$ and $a \in \text{Mor}_{A_{\mathfrak{F}_2}}(1, 7)$. Define

$$\mathcal{D} := \bigcup_{n \in \mathbb{N}} \text{Mor}_{A_{\mathfrak{F}_k}}(1, n)$$

as the set of all annular k -trees, and denote $\ell(f) := \text{target}(f)$ for $f \in \mathcal{D}$.

Fraction notation

Define \sim on pairs of annular k -trees as $(a, y) \sim (b, z)$ if and only if there exist $r, s \in \text{Mor}_{A\tilde{\mathfrak{F}}_k}$ such that $(r \circ a, r \circ y) = (s \circ b, s \circ z)$

Proposition (Brown 1987)

Two pairs of annular k -trees (a, y) and (b, z) correspond to the same element in T_k if and only if $(a, y) \sim (b, z)$.

Denote by $[(c, x)] \equiv \frac{c}{x}$ the equivalence class (c, x)

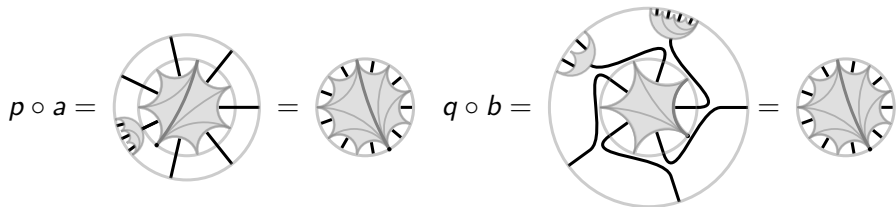
$$\frac{p \circ c}{p \circ x} = \frac{c}{x},$$

where we interpret $p \in \text{Mor}_{A\tilde{\mathfrak{F}}_k}$ as being 'cancelled' in the fraction. Taking the trees from a previous slide:

$$\frac{\begin{array}{c} \text{Tree 1} \\ \text{Tree 2} \end{array}}{\begin{array}{c} \text{Tree 3} \\ \text{Tree 4} \end{array}} = \frac{\begin{array}{c} \text{Tree 1} \\ \text{Tree 5} \end{array}}{\begin{array}{c} \text{Tree 3} \\ \text{Tree 6} \end{array}} = \frac{\begin{array}{c} \text{Tree 1} \\ \text{Tree 7} \end{array}}{\text{Tree 4}}$$

Composition via fractions

Any $a, b \in \mathcal{D}$ admit $p, q \in \text{Mor}_{A\tilde{\mathcal{S}}_k}$ such that $p \circ a = q \circ b$, called a *stabilisation*:



Composition of functions can be expressed as the product of fractions

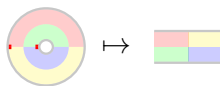
$$G \circ H = \frac{a}{b} \frac{c}{d} = \frac{\cancel{q \circ a}}{q \circ b} \frac{p \circ c}{\cancel{p \circ d}} = \frac{p \circ c}{q \circ b}, \quad \text{where } q \circ a = p \circ d,$$

where $G \circ H$ has the domain of H and the range of G .

Representations of T_k

Preliminaries

For convenience we will use the cutting convention:



Denote by Hilb the category of Hilbert spaces:

- Obj_{Hilb} are Hilbert spaces V_n for each $n \in \mathbb{N}$
- $\text{Mor}_{\text{Hilb}}(V_m, V_n)$ are linear maps

where the inner product on each V_n can be expressed diagrammatically as:

$$\langle x, y \rangle_n = \begin{array}{c} \boxed{x^*} \\ | \quad | \quad | \quad \dots \\ \boxed{y} \end{array}, \quad \begin{array}{c} \overbrace{\quad \quad \quad}^n \\ | \quad | \quad | \quad \dots \\ \boxed{x} \end{array}, \quad \begin{array}{c} \overbrace{\quad \quad \quad}^n \\ | \quad | \quad | \quad \dots \\ \boxed{y} \end{array} \in V_n$$

Jones' action

Define the functor $\Phi : A\mathfrak{S}_k \rightarrow \text{Hilb}$

- $\Phi_0(n) = V_n$ for all $n \in \mathbb{N}$
- $\Phi_1^R(p) \in \text{Mor}_{\text{Hilb}}(V_m, V_n)$ for all $p \in \text{Mor}_{A\mathfrak{S}_k}(m, n)$

$$p = \text{[diagram]}, \quad \Phi_1^R(p) = \text{[diagram]}, \quad \text{[diagram]} \in P_{k+1}$$

Construct the set A_Φ such that:

	T_k	A_Φ
Elements	$\frac{f}{g} = \frac{p \circ f}{p \circ g}$	$\frac{f}{x} = \frac{p \circ f}{\Phi_1^R(p)(x)}$
Action of T_k	$\frac{f_1}{g_1} \frac{f_2}{g_2} = \frac{q \circ f_2}{p \circ g_1}$	$\frac{f_1}{x_1} \frac{f_2}{g_2} = \frac{p \circ f_1}{\Phi_1^R(p)(x_1)} \frac{q \circ f_2}{q \circ g_2} = \frac{q \circ f_2}{\Phi_1^R(p)(x_1)}$

where $x \in V_{\ell(f)}$, $x_1 \in V_{\ell(f_1)}$. Inducing Hilbert space \mathfrak{H} and a representation

$$\pi_R : T_k \rightarrow \text{End}(A_\Phi).$$

Continuity conditions

Continuous representations

Definition

A representation π is *continuous* if each sequence $(f_n)_{n \in \mathbb{N}} \subset T_k$ satisfies

$$\lim_{n \rightarrow \infty} \|f_n - \text{id}\| = 0, \quad \lim_{n \rightarrow \infty} \langle \xi, \pi(f_n)(\eta) \rangle = \langle \xi, \eta \rangle, \quad \forall \xi, \eta \in \mathfrak{H}.$$

Denote by Rot_k the rotation subgroup of T_k , generated by:

$$\varrho_s : S^1 \rightarrow S^1, \quad x \mapsto x + s \pmod{1},$$

where s is a k -adic rational. Matrix elements can be expressed as:

$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{kr}}) \left(\frac{i}{y} \right) \right\rangle = \frac{\overbrace{\dots}^{x^*}}{\underbrace{\dots}_y} = \langle x, \Omega_{kr} y \rangle_{kr}, \quad \Omega_n := \underbrace{\overbrace{\dots}^n}_n$$

where $x, y \in V_{kr}$.

The matrix elements can be expressed in terms of a *transfer operator*:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{k^{r+s}}}) \left(\frac{i}{y} \right) \right\rangle = \langle x, T_{kr}(\mathfrak{R}^s(v))y \rangle_{kr}, \quad v := \text{---} \circlearrowright \text{---}$$

where $x, y \in V_{kr}$, and

$$T_n(a) = \underbrace{\text{---} \overset{\circlearrowright}{a} \text{---} \overset{\circlearrowright}{a} \text{---} \dots \overset{\circlearrowright}{a} \text{---}}_n$$

$$\mathfrak{R}(a) = \text{---} \overset{\circlearrowright}{a} \text{---} \overset{\circlearrowright}{a} \text{---} \dots \overset{\circlearrowright}{a} \text{---}$$

The diagram for $T_n(a)$ shows a horizontal chain of n nodes, each labeled a with a red dot and a clockwise rotation arrow. The nodes are connected by horizontal lines. Above and below the chain are two parallel horizontal lines, and vertical lines connect the nodes to these lines, forming a series of rectangles. The diagram for $\mathfrak{R}(a)$ shows a single node a with a red dot and a clockwise rotation arrow, connected to a hexagonal lattice structure. The top and bottom vertices of the hexagon are labeled R^* and R respectively. The lattice extends horizontally to the right, with ellipses indicating continuation.

To illustrate, we present a small example where $(k, r, s) = (3, 1, 2)$:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{3^3}}) \left(\frac{i}{y} \right) \right\rangle =$$

The diagram shows a 3x3 grid of hexagonal lattices. The top and bottom rows are shaded light purple and labeled X^* and y respectively. Each hexagon has vertices labeled R^* and R . The lattices are connected horizontally and vertically, forming a grid. A dashed line with a red dot and a clockwise rotation arrow is drawn across the middle row of hexagons, representing the operator $\pi_R(\varrho_{\frac{1}{3^3}})$.

The matrix elements can be expressed in terms of a *transfer operator*:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{k^{r+s}}}) \left(\frac{i}{Y} \right) \right\rangle = \langle x, T_{kr}(\mathfrak{R}^s(v))y \rangle_{kr}, \quad v := \text{---} \circlearrowleft \text{---}$$

where $x, y \in V_{kr}$, and

$$T_n(a) = \underbrace{\text{---} \overset{\circlearrowleft}{a} \text{---} \overset{\circlearrowleft}{a} \text{---} \dots \text{---} \overset{\circlearrowleft}{a} \text{---}}_n$$

$$\mathfrak{R}(a) = \text{---} \overset{\circlearrowleft}{a} \text{---} \overset{\circlearrowleft}{a} \text{---} \dots \text{---} \overset{\circlearrowleft}{a} \text{---} \text{---}$$

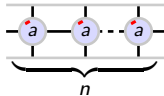
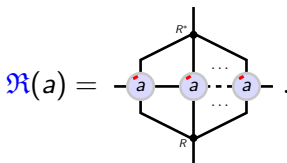
To illustrate, we present a small example where $(k, r, s) = (3, 1, 2)$:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{3^3}}) \left(\frac{i}{Y} \right) \right\rangle = \text{---} \overset{\circlearrowleft}{v} \text{---} \overset{\circlearrowleft}{v} \text{---} \dots \text{---} \overset{\circlearrowleft}{v} \text{---} \text{---}$$

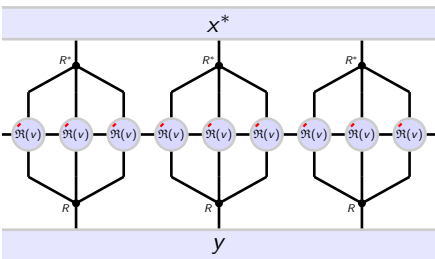
The matrix elements can be expressed in terms of a *transfer operator*:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{k^{r+s}}}) \left(\frac{i}{Y} \right) \right\rangle = \left\langle X, T_{kr}(\mathfrak{R}^s(v)) Y \right\rangle_{kr}, \quad v := \text{---} \circlearrowleft \text{---}$$

where $x, y \in V_{kr}$, and

$$T_n(a) = \text{---} \underbrace{\text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---}}_n \text{---} \quad \mathfrak{R}(a) = \text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---}$$



To illustrate, we present a small example where $(k, r, s) = (3, 1, 2)$:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{3^3}}) \left(\frac{i}{Y} \right) \right\rangle = \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---}$$


The matrix elements can be expressed in terms of a *transfer operator*:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{kr+s}}) \left(\frac{i}{y} \right) \right\rangle = \left\langle x, T_{kr}(\mathfrak{R}^s(v))y \right\rangle_{kr}, \quad v := \text{---} \circlearrowleft \text{---}$$

where $x, y \in V_{kr}$, and

$$T_n(a) = \underbrace{\text{---} \circlearrowleft \text{---} \text{---} \circlearrowleft \text{---} \text{---} \circlearrowleft \text{---}}_n$$

$$\mathfrak{R}(a) = \begin{array}{c} R^* \\ \diagdown \quad \diagup \\ \text{---} \circlearrowleft \text{---} \quad \text{---} \circlearrowleft \text{---} \quad \dots \\ \diagup \quad \diagdown \\ R \end{array}$$

To illustrate, we present a small example where $(k, r, s) = (3, 1, 2)$:

$$\left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{33}}) \left(\frac{i}{y} \right) \right\rangle = \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \text{---} \circlearrowleft \text{---} \quad \text{---} \circlearrowleft \text{---} \quad \text{---} \circlearrowleft \text{---} \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \end{array}$$

x^*

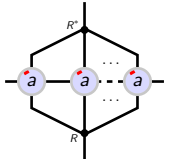
y

The matrix elements can be expressed in terms of a *transfer operator*:

$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{kr+s}}) \left(\frac{i}{y} \right) \right\rangle = \left\langle x, T_{kr}(\mathfrak{R}^s(v))y \right\rangle_{kr}, \quad v := \text{---} \circlearrowleft \text{---}$$

where $x, y \in V_{kr}$, and

$$T_n(a) = \text{---} \underbrace{\text{---} \circlearrowleft \text{---} \circlearrowleft \text{---} \circlearrowleft \text{---}}_n \text{---}$$

$$\mathfrak{R}(a) = \text{---} \circlearrowleft \text{---} \circlearrowleft \text{---} \circlearrowleft \text{---} \text{---}$$


To illustrate, we present a small example where $(k, r, s) = (3, 1, 2)$:

$$\left\langle \frac{i}{x}, \pi_R(\varrho_{\frac{1}{3^3}}) \left(\frac{i}{y} \right) \right\rangle = \left\langle x, T_3(\mathfrak{R}^2(v))y \right\rangle_3$$

Continuity conditions

Applying:

$$\lim_{s \rightarrow \infty} \left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{kr+s}}) \left(\frac{i}{y} \right) \right\rangle = \lim_{s \rightarrow \infty} \left\langle x, T_{kr}(\mathfrak{R}^s(v))y \right\rangle_{kr} \stackrel{(!)}{=} \langle x, y \rangle_{kr}$$

Proposition

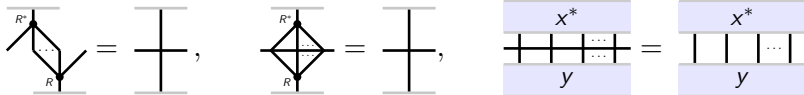
If there exists a $w \in P_4$ such that

$$\mathfrak{R}(v) = w, \quad \mathfrak{R}(w) = w, \quad \langle x, T_{kr}(w)y \rangle_{kr} = \langle x, y \rangle_{kr} \quad (\star)$$

for all $x, y \in V_{kr}$, then

$$\lim_{s \rightarrow \infty} \left\langle \frac{i}{X}, \pi_R(\varrho_{\frac{1}{kr+s}}) \left(\frac{i}{y} \right) \right\rangle = \langle x, y \rangle_{kr}.$$

A concrete realisation of (\star) is given by:



Brauer algebra solution

The Brauer planar algebra $(B_n)_{n \in 2\mathbb{N}_0}$ is generated by the action of planar tangles on the space $B_4 = \text{span}(\{ \textcircled{\bullet} \textcircled{} \}, \textcircled{\smile} \textcircled{} \}, \textcircled{\times} \textcircled{} \})$, subject to:

$$\textcircled{\times} = \textcircled{\bullet} \textcircled{} \quad \textcircled{\mathcal{D}} = \delta \textcircled{} \quad \textcircled{\text{loop}} = \textcircled{\smile}$$

Specialising $k = 5$ and $\delta = 1$ we have the solution $\textcircled{R} = \textcircled{\Psi}$

$$\textcircled{R^*} = \textcircled{\text{loop}} = \textcircled{\text{cross}}, \quad \textcircled{R^*} = \textcircled{\text{loop}} = \textcircled{\text{cross}}.$$

This solution can be generalised to $k = 2n + 5$ for all $n \in \mathbb{N}_0$

$$\textcircled{R} = \textcircled{\Psi} \textcircled{\text{loop}}, \quad \textcircled{\text{loop}} \in P_{2n}.$$

Theorem

For $R \in P_{2n+5}$ above, π_R is a continuous unitary representation of Rot_{2n+5} .

Brauer algebra solution

The Brauer planar algebra $(B_n)_{n \in 2\mathbb{N}_0}$ is generated by the action of planar tangles on the space $B_4 = \text{span}(\{ \bullet \circlearrowleft, \bullet \circlearrowright, \bullet \otimes \bullet \})$, subject to:

$$\bullet \otimes \bullet = \bullet \circlearrowleft \quad \bullet \circlearrowright = \delta \bullet \circlearrowleft \quad \bullet \circlearrowright = \bullet \circlearrowleft$$

Specialising $k = 5$ and $\delta = 1$ we have the solution $\begin{matrix} R^* \\ \vee \\ R \end{matrix} = \begin{matrix} \vee \\ \cup \end{matrix}$

$$\begin{matrix} R^* \\ \text{diamond} \\ R \end{matrix} = \begin{matrix} \text{blue circle} \\ \text{red circle} \end{matrix} = \begin{matrix} \text{red line} \\ \text{blue line} \end{matrix}, \quad \begin{matrix} R^* \\ \text{diamond} \\ R \end{matrix} = \begin{matrix} \text{blue circle} \\ \text{red circle} \end{matrix} = \begin{matrix} \text{red line} \\ \text{blue line} \end{matrix}.$$

This solution can be generalised to $k = 2n + 5$ for all $n \in \mathbb{N}_0$

$$\begin{matrix} R^* \\ \vee \\ R \end{matrix} = \begin{matrix} \vee \\ \cup \\ \text{blue circle } n \end{matrix}, \quad \text{blue circle } n \in P_{2n}.$$

Theorem

For $R \in P_{2n+5}$ above, π_R is a continuous unitary representation of Rot_{2n+5} .

Outlook

Outlook

Summary:

- Semicontinuous models of conformal nets via planar algebras.
- Limited by the continuity of the representations of T_k .
- Developed sufficient conditions that imply continuity of representations of the rotation subgroup of T_k .

Future work:

- Solve the continuity conditions for other types of planar algebras.
- Construct sufficient conditions that implies the continuity of representation of all T_k
- Develop the limit that takes continuous representations of T_k to continuous representations of $\text{Diff}_+(S^1)$.

The end!