

# Subfactors and quantum integrability

Xavier Poncini

University of Queensland



University of Melbourne Mathematical Physics Seminar

*Based on work (arXiv:2206.14462 + arXiv:2302.11712) with Jørgen Rasmussen*

# Motivation

**Subfactors** appear in many interacting areas of mathematical physics:

- tensor categories
- quantum groups
- vertex operators algebras
- quantum field theory
- knot theory
- statistical mechanics – **quantum integrability**

## Question

Which **subfactors** encode the structure of **quantum integrable models**?

We will encounter subfactors in their incarnation as **planar algebras**.

# Outline

- 1 Subfactors
- 2 Quantum integrability
- 3 Planar-algebraic models
- 4 Outlook

## Subfactors

# Subfactors

A **factor** is an infinite-dimensional von Neumann algebra with **trivial centre** and a **trace**. For two factors  $M_0 \subset M_1$ ,  $M_0$  is a **subfactor** of  $M_1$ .

Further subfactors can be produced via **Jones' basic construction**:

$$M_0 \subset M_1 \subset^{e_1} M_2 \subset^{e_2} M_3 \subset \dots \quad \text{where } M_{k+1} := \langle M_k, e_k \rangle$$

and  $\langle e_1, e_2, \dots, e_n \rangle$  is a finite-dim.  $C^*$ -algebra subject to the relations:

$$e_i^2 = e_i = e_i^*, \quad e_i e_{i\pm 1} e_i = \delta^{-2} e_i, \quad e_i e_j = e_j e_i, \quad |i - j| \geq 2$$

This is the **Temperley–Lieb (TL) algebra** which can be expressed:

$$e_i \leftrightarrow \frac{1}{\delta} \begin{array}{|c|} \hline \dots & \text{---} & \text{---} & \dots \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \dots & \text{---} & \text{---} & \dots \\ \hline \end{array}, \quad \text{e.g.} \quad \begin{array}{|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} = \delta \begin{array}{|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array}$$

# The standard invariant

The **standard invariant** of a subfactor consists of the two **towers**:

$$\begin{array}{ccccccc}
 Z(M_0) = & M'_0 \cap M_0 & \subset & M'_0 \cap M_1 & \subset & M'_0 \cap M_2 & \subset & \dots \\
 & & & \cup & & \cup & & \\
 & & & Z(M_1) = & M'_1 \cap M_1 & \subset & M'_1 \cap M_2 & \subset & \dots
 \end{array}$$

where

$$M'_n \cap M_m = \{x \in M_m \mid xy = yx, \forall y \in M_n\}, \quad n \in \{0, 1\}$$

are finite dimensional  $C^*$ -algebras that include Temperley-Lieb algebras.

Thanks to a theorem of Popa, a subfactor can be **reconstructed** from the standard invariant. The standard invariant **stores** the data of a subfactor.

**Planar algebras** provide a pictorial description of the standard invariant.

# The standard invariant

The **standard invariant** of a subfactor consists of the two **towers**:

$$\begin{array}{ccccccc} \mathbb{C} & = & M'_0 \cap M_0 & \subset & M'_0 \cap M_1 & \subset & M'_0 \cap M_2 & \subset & \dots \\ & & & & \cup & & \cup & & \\ & & \mathbb{C} & = & M'_1 \cap M_1 & \subset & M'_1 \cap M_2 & \subset & \dots \end{array}$$

where

$$M'_n \cap M_m = \{x \in M_m \mid xy = yx, \forall y \in M_n\}, \quad n \in \{0, 1\}$$

are finite dimensional  $C^*$ -algebras that include Temperley-Lieb algebras.

Thanks to a theorem of Popa, a subfactor can be **reconstructed** from the standard invariant. The standard invariant **stores** the data of a subfactor.

**Planar algebras** provide a pictorial description of the standard invariant.

# Planar algebras

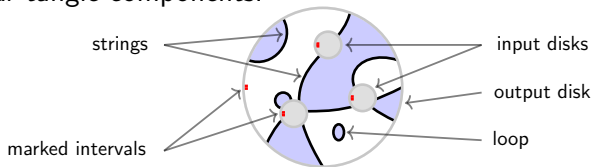
## Definition

A **planar algebra** is a collection of vector spaces  $(A_{n,\pm})_{n \in \mathbb{N}_0}$ , together with the action of **shaded planar tangles** as multilinear maps e.g.

$$T = \text{[Diagram of a shaded planar tangle with three input disks labeled 1, 2, 3 and one output disk labeled 0]} , \quad P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

such that this action is compatible with the composition of tangles.

Shaded planar tangle components:



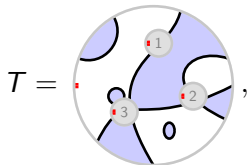
Planar algebras naturally describe interactions in two-dimensions!



# Planar algebras

## Definition

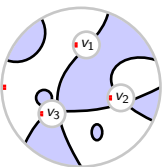
A **planar algebra** is a collection of vector spaces  $(A_{n,\pm})_{n \in \mathbb{N}_0}$ , together with the action of **shaded planar tangles** as multilinear maps e.g.



$$P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

such that this action is compatible with the composition of tangles.

Multilinear map action:

$$P_T(v_1, v_2, v_3) =$$


$$\in A_{4,+}$$

Planar algebras naturally describe interactions in two-dimensions!

# Planar algebras

## Definition

A **planar algebra** is a collection of vector spaces  $(A_{n,\pm})_{n \in \mathbb{N}_0}$ , together with the action of **shaded planar tangles** as multilinear maps e.g.

$$T = \text{[Diagram of a shaded planar tangle with 4 inputs and 1 output]} , \quad P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

such that this action is compatible with the composition of tangles.

Composition of tangles:

$$T = \text{[Diagram of tangle T]} , \quad S = \text{[Diagram of tangle S]} , \quad T \circ_D S = \text{[Diagram of the composition of T and S]}$$

Planar algebras naturally describe interactions in two-dimensions!

# Planar algebras

## Definition

A **planar algebra** is a collection of vector spaces  $(A_{n,\pm})_{n \in \mathbb{N}_0}$ , together with the action of **shaded planar tangles** as multilinear maps e.g.

$$T = \text{[Diagram of a shaded planar tangle with 4 inputs and 1 output]} , \quad P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

such that this action is compatible with the composition of tangles.

Composition of tangles:

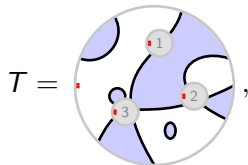
$$T = \text{[Diagram of tangle T]} , \quad S = \text{[Diagram of tangle S]} , \quad T \circ_D S = \text{[Diagram of the composition of T and S]}$$

Planar algebras naturally describe interactions in two-dimensions!

# Planar algebras

## Definition

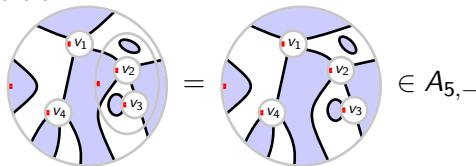
A **planar algebra** is a collection of vector spaces  $(A_{n,\pm})_{n \in \mathbb{N}_0}$ , together with the action of **shaded planar tangles** as multilinear maps e.g.



$$P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

such that this action is compatible with the composition of tangles.

Compatibility condition:



Planar algebras naturally describe interactions in two-dimensions!

# Planar algebras

A planar algebra contains countably many **associative algebras**:

$$M_{n,+} := \text{Diagram 1}, \quad M_{n,-} := \text{Diagram 2},$$

with  $M_{n,\pm}$  inducing a multiplication on  $A_{n,\pm}$  e.g.

$$\mathbf{xy} = \text{Diagram 3} = P_{M_{n,+}}(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in A_{n,+}.$$

Under mild conditions the algebras are **unital** with units

$$\mathbf{1}_{n,\pm} := P_{\text{Id}_{n,\pm}}(), \quad \text{Id}_{n,+} := \text{Diagram 4}, \quad \text{Id}_{n,-} := \text{Diagram 5}.$$

# Example: Temperley-Lieb planar algebra

## Vector spaces:

$$A_{0,+} = \text{span}\left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \end{array} \right\}, \quad A_{1,+} = \text{span}\left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \end{array} \right\}, \quad A_{2,+} = \text{span}\left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \text{---} \end{array} \right\}, \quad \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \end{array}$$

$$A_{0,-} = \text{span}\left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \end{array} \right\}, \quad A_{1,-} = \text{span}\left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \end{array} \right\}, \quad A_{2,-} = \text{span}\left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \text{---} \end{array} \right\}, \quad \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \end{array}$$

...

## Planar tangle action:

$$P_T: A_{2,+} \times A_{2,+} \rightarrow A_{3,+}$$

$$T = \begin{array}{c} \text{red dot} \\ \bigcirc \end{array}, \quad P_T\left( \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \end{array}, \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \end{array} \right) = \begin{array}{c} \text{red dot} \\ \bigcirc \text{---} \text{---} \end{array} = \delta \begin{array}{c} \text{red dot} \\ \bigcirc \end{array}$$

Every planar algebra contains Temperley-Lieb-like planar algebra!

# Example: Temperley-Lieb planar algebra

## Vector spaces:

$$\begin{aligned}
 A_{0,+} &= \text{span} \left\{ \text{white circle with red dot} \right\}, & A_{1,+} &= \text{span} \left\{ \text{white circle with red dot and vertical line} \right\}, & A_{2,+} &= \text{span} \left\{ \text{white circle with red dot and two arcs (top)}, \text{white circle with red dot and two arcs (bottom)} \right\} \\
 A_{0,-} &= \text{span} \left\{ \text{shaded circle with red dot} \right\}, & A_{1,-} &= \text{span} \left\{ \text{shaded circle with red dot and vertical line} \right\}, & A_{2,-} &= \text{span} \left\{ \text{shaded circle with red dot and two arcs (top)}, \text{shaded circle with red dot and two arcs (bottom)} \right\}
 \end{aligned}$$

...

## Planar tangle action:

$$T = \text{white circle with red dot and two arcs (top and bottom)}, \quad P_T: A_{2,+} \times A_{2,+} \rightarrow A_{3,+}$$

$$P_T \left( \text{white circle with red dot and two arcs (top)}, \text{white circle with red dot and two arcs (bottom)} \right) = \text{white circle with red dot and two arcs (top and bottom)} = \delta \cdot \text{white circle with red dot}$$

The shading of a planar algebra need not carry any non-trivial information. In this case, it can be ignored and the planar algebra is called **unshaded**.

# Example: Temperley-Lieb planar algebra

## Vector spaces:

$$A_0 = \text{span} \left\{ \begin{array}{c} \bullet \\ \bigcirc \end{array} \right\}, \quad A_1 = \text{span} \left\{ \begin{array}{c} \bullet \\ \bigcirc \mid \end{array} \right\}, \quad A_2 = \text{span} \left\{ \begin{array}{c} \bullet \\ \bigcirc \mid \bigcirc \end{array}, \begin{array}{c} \bullet \\ \bigcirc \mid \bigcirc \end{array} \right\},$$

...

## Planar tangle action:

$$P_T: A_2 \times A_2 \rightarrow A_3$$

$$T = \begin{array}{c} \bullet \\ \bigcirc \end{array}, \quad P_T \left( \begin{array}{c} \bullet \\ \bigcirc \mid \end{array}, \begin{array}{c} \bullet \\ \bigcirc \mid \bigcirc \end{array} \right) = \begin{array}{c} \bullet \\ \bigcirc \mid \bigcirc \end{array} = \delta \begin{array}{c} \bullet \\ \bigcirc \mid \bigcirc \end{array}$$

The shading of a planar algebra need not carry any non-trivial information. In this case, it can be ignored and the planar algebra is called **unshaded**.



# Example: Tensor planar algebra

## Vector spaces:

$$A_{0,\pm} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \bigcirc \end{array} \right\}, \quad A_{1,\pm} = \text{span} \left\{ \begin{array}{c} i \\ \text{red square} \bigcirc \\ j \end{array} \mid i, j = 1, \dots, N \right\},$$

$$A_{2,\pm} = \text{span} \left\{ \begin{array}{c} i \quad j \\ \text{red square} \bigcirc \\ l \quad k \end{array} \mid i, j, k, l = 1, \dots, N \right\}, \quad \dots$$

## Planar tangle action:

$$P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

$$T = \begin{array}{c} \text{red square} \\ \bigcirc \\ \text{blue shaded region with black lines and circles} \\ \text{circles labeled 1, 2, 3} \\ \text{circle labeled 0} \end{array}, \quad P_T \left( \begin{array}{c} 1 \\ \text{red square} \bigcirc \\ 3 \end{array}, \begin{array}{c} 3 \quad 4 \\ \text{red square} \bigcirc \\ 2 \quad 3 \end{array}, \begin{array}{c} 1 \quad 3 \\ \text{red square} \bigcirc \\ 2 \quad 4 \end{array} \right) = \begin{array}{c} \text{red square} \\ \bigcirc \\ \text{blue shaded region with black lines and circles} \\ \text{circles labeled 1, 2, 3, 4} \\ \text{circle labeled 0} \end{array}$$

# Example: Tensor planar algebra

**Vector spaces:**

$$A_{0,\pm} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \bigcirc \end{array} \right\}, \quad A_{1,\pm} = \text{span} \left\{ \begin{array}{c} i \\ \text{red square} \\ \bigcirc \\ j \end{array} \mid i, j = 1, \dots, N \right\},$$

$$A_{2,\pm} = \text{span} \left\{ \begin{array}{c} i \quad j \\ \text{red square} \\ \bigcirc \\ l \quad k \end{array} \mid i, j, k, l = 1, \dots, N \right\}, \quad \dots$$

**Planar tangle action:**

$$P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

$$T = \begin{array}{c} \text{red square} \\ \bigcirc \\ \text{blue disk with black lines and circles} \\ \text{red square} \end{array}, \quad P_T \left( \begin{array}{c} 1 \\ \text{red square} \\ \bigcirc \\ 3 \end{array}, \begin{array}{c} 3 \quad 4 \\ \text{red square} \\ \bigcirc \\ 2 \quad 3 \end{array}, \begin{array}{c} 1 \quad 3 \\ \text{red square} \\ \bigcirc \\ 2 \quad 4 \end{array} \right) = \sum_{i,j=1}^N \begin{array}{c} i \quad 1 \\ \text{red square} \\ \bigcirc \\ 2 \quad j \end{array}$$

# Example: Tensor planar algebra

## Vector spaces:

$$A_{0,\pm} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \bigcirc \end{array} \right\}, \quad A_{1,\pm} = \text{span} \left\{ \begin{array}{c} i \\ \text{red square} \bigcirc \\ j \end{array} \mid i, j = 1, \dots, N \right\},$$

$$A_{2,\pm} = \text{span} \left\{ \begin{array}{c} i \quad j \\ \text{red square} \bigcirc \\ l \quad k \end{array} \mid i, j, k, l = 1, \dots, N \right\}, \quad \dots$$

## Planar tangle action:

$$P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

$$T = \begin{array}{c} \text{red square} \\ \bigcirc \\ \text{tangle} \end{array}, \quad P_T \left( \begin{array}{c} 1 \\ \text{red square} \bigcirc \\ 3 \end{array}, \begin{array}{c} 3 \quad 4 \\ \text{red square} \bigcirc \\ 2 \quad 3 \end{array}, \begin{array}{c} 1 \quad 3 \\ \text{red square} \bigcirc \\ 2 \quad 4 \end{array} \right) = \sum_{i,j=1}^N \begin{array}{c} i \quad 1 \\ \text{red square} \bigcirc \\ 2 \quad 3 \quad 4 \end{array}$$

# Example: Tensor planar algebra

## Vector spaces:

$$A_{0,\pm} = \text{span} \left\{ \begin{array}{c} \text{red dot} \\ \bigcirc \end{array} \right\}, \quad A_{1,\pm} = \text{span} \left\{ \begin{array}{c} i \\ \text{red dot} \\ \bigcirc \\ j \end{array} \mid i, j = 1, \dots, N \right\},$$

$$A_{2,\pm} = \text{span} \left\{ \begin{array}{c} i \quad j \\ \text{red dot} \\ \bigcirc \\ k \end{array} \mid i, j, k, l = 1, \dots, N \right\}, \quad \dots$$

## Planar tangle action:

$$P_T : A_{1,+} \times A_{2,-} \times A_{3,+} \rightarrow A_{4,+}$$

$$T = \begin{array}{c} \text{red dot} \\ \bigcirc \\ \text{shaded regions with labels 1, 2, 3, 0} \end{array}, \quad P_T \left( \begin{array}{c} 1 \\ \text{red dot} \\ \bigcirc \\ 3 \end{array}, \begin{array}{c} 3 \quad 4 \\ \text{red dot} \\ \bigcirc \\ 2 \quad 3 \end{array}, \begin{array}{c} 1 \quad 3 \\ \text{red dot} \\ \bigcirc \\ 2 \quad 4 \end{array} \right) = N \sum_{i=1}^N \begin{array}{c} i \quad 1 \\ \text{red dot} \\ \bigcirc \\ 2 \quad 4 \\ 4 \quad 3 \end{array}$$

The shading of a planar algebra need not carry any non-trivial information. In this case, it can be ignored and the planar algebra is called **unshaded**.

# Example: Tensor planar algebra

## Vector spaces:

$$A_0 = \text{span} \left\{ \begin{array}{c} \bullet \\ \circ \end{array} \right\}, \quad A_1 = \text{span} \left\{ \begin{array}{c} i \\ \bullet \\ \circ \\ \bullet \\ j \end{array} \right\} \mid i, j = 1, \dots, N \Big\},$$

$$A_2 = \text{span} \left\{ \begin{array}{c} i \quad j \\ \bullet \quad \bullet \\ \circ \\ \bullet \quad \bullet \\ l \quad k \end{array} \right\} \mid i, j, k, l = 1, \dots, N \Big\}, \quad \dots$$

## Planar tangle action:

$$P_T : A_1 \times A_2 \times A_3 \rightarrow A_4$$

$$T = \begin{array}{c} \bullet \\ \circ \end{array}, \quad P_T \left( \begin{array}{c} 1 \\ \bullet \\ \circ \\ \bullet \\ 3 \end{array}, \begin{array}{c} 3 \quad 4 \\ \bullet \quad \bullet \\ \circ \\ \bullet \quad \bullet \\ 2 \quad 3 \end{array}, \begin{array}{c} 1 \quad 3 \\ \bullet \quad \bullet \\ \circ \\ \bullet \quad \bullet \\ 2 \quad 4 \end{array} \right) = N \sum_{i=1}^N \begin{array}{c} i \quad 1 \\ \bullet \quad \bullet \\ \circ \\ \bullet \quad \bullet \\ 2 \quad 4 \\ 4 \quad 3 \end{array}$$

The shading of a planar algebra need not carry any non-trivial information. In this case, it can be ignored and the planar algebra is called **unshaded**.

# Subfactor planar algebras

## Definition

A planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  that is involutive, evaluable, spherical and positive-definite is called a **subfactor planar algebra**.

There is an inner product on each  $A_{n,\pm}$ , for example:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{2,+} := \text{Diagram}$$

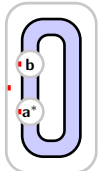
With multiplication tangle, each  $A_{n,\pm}$  is a finite-dimensional  $C^*$ -algebra with a Temperley-Lieb subalgebra.

# Subfactor planar algebras

## Definition

A planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  that is **involutive**, evaluable, spherical and positive-definite is called a **subfactor planar algebra**.

There is an inner product on each  $A_{n,\pm}$ , for example:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{2,+} := \left( \text{Diagram} \right), \quad \mathbf{a}^* \text{ well-defined}$$


With multiplication tangle, each  $A_{n,\pm}$  is a finite-dimensional  $C^*$ -algebra with a Temperley-Lieb subalgebra.

# Subfactor planar algebras

## Definition

A planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  that is involutive, **evaluable**, spherical and positive-definite is called a **subfactor planar algebra**.

There is an inner product on each  $A_{n,\pm}$ , for example:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{2,+} := \left( \text{Diagram} \right) \in \mathbb{C}, \quad \dim(A_{n,\pm}) < \infty$$

With multiplication tangle, each  $A_{n,\pm}$  is a finite-dimensional  $C^*$ -algebra with a Temperley-Lieb subalgebra.



# Subfactor planar algebras

## Definition

A planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  that is involutive, evaluable, **spherical** and positive-definite is called a **subfactor planar algebra**.

There is an inner product on each  $A_{n,\pm}$ , for example:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{2,+} := \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}$$

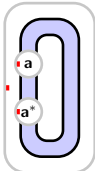
With multiplication tangle, each  $A_{n,\pm}$  is a finite-dimensional  $C^*$ -algebra with a Temperley-Lieb subalgebra.

# Subfactor planar algebras

## Definition

A planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  that is involutive, evaluable, spherical and **positive-definite** is called a **subfactor planar algebra**.

There is an inner product on each  $A_{n,\pm}$ , for example:

$$\langle \mathbf{a}, \mathbf{a} \rangle_{2,+} := \text{[Diagram of a cup with elements } \mathbf{a} \text{ and } \mathbf{a}^* \text{]} > 0$$


The diagram shows a light blue rounded rectangle containing a white cup-shaped tangle. Two small white circles, labeled 'a' and 'a\*', are positioned on the left side of the cup, connected by a vertical line. The 'a' circle is at the top and the 'a\*' circle is at the bottom. To the right of the cup is a greater-than sign '>' followed by a zero '0'.

With multiplication tangle, each  $A_{n,\pm}$  is a finite-dimensional  $C^*$ -algebra with a Temperley-Lieb subalgebra.

# Relation to subfactors

Recall the standard invariant of a subfactor  $M_0 \subset M_1$ :

$$\begin{array}{ccccccc} \mathbb{C} & = & M'_0 \cap M_0 & \subset & M'_0 \cap M_1 & \subset & M'_0 \cap M_2 & \subset & \dots \\ & & & & \cup & & \cup & & \dots \\ & & \mathbb{C} & = & M'_1 \cap M_1 & \subset & M'_1 \cap M_2 & \subset & \dots \end{array}$$

The standard invariant **is** a subfactor planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  where

$$A_{k,+} = M'_0 \cap M_k, \quad A_{k,-} = M'_1 \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

$$P \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k-1,+}$$

$$P \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k+1,+}$$

$$P \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k-1,-}$$

$$P \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,-} \rightarrow A_{k+1,+}$$

# Relation to subfactors

Recall the standard invariant of a subfactor  $M_0 \subset M_1$ :

$$\begin{array}{ccccccc} \mathbb{C} & = & A_{0,+} & \subset & A_{1,+} & \subset & A_{2,+} & \subset & \dots \\ & & & & \cup & & \cup & & \dots \\ & & \mathbb{C} & = & A_{0,-} & \subset & A_{1,-} & \subset & \dots \end{array}$$

The standard invariant **is** a subfactor planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  where

$$A_{k,+} = M'_0 \cap M_k, \quad A_{k,-} = M'_1 \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

$$P \quad \text{[Diagram]} : A_{k,+} \rightarrow A_{k-1,+}$$

$$P \quad \text{[Diagram]} : A_{k,+} \rightarrow A_{k+1,+}$$

$$P \quad \text{[Diagram]} : A_{k,+} \rightarrow A_{k-1,-}$$

$$P \quad \text{[Diagram]} : A_{k,-} \rightarrow A_{k+1,+}$$

# Relation to subfactors

Recall the standard invariant of a subfactor  $M_0 \subset M_1$ :

$$\mathbb{C} = A_{0,+} \leftarrow A_{1,+} \leftarrow A_{2,+} \leftarrow \dots$$

$$\cup \qquad \cup \qquad \dots$$

$$\mathbb{C} = A_{0,-} \subset A_{1,-} \subset \dots$$

The standard invariant **is** a subfactor planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  where

$$A_{k,+} = M'_0 \cap M_k, \qquad A_{k,-} = M'_1 \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k-1,+}$$

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k+1,+}$$

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k-1,-}$$

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,-} \rightarrow A_{k+1,+}$$

# Relation to subfactors

Recall the standard invariant of a subfactor  $M_0 \subset M_1$ :

$$\begin{array}{ccccccc}
 \mathbb{C} & = & A_{0,+} & \rightarrow & A_{1,+} & \rightarrow & A_{2,+} & \rightarrow & \dots \\
 & & & & \cup & & \cup & & \dots \\
 & & \mathbb{C} & = & A_{0,-} & \subset & A_{1,-} & \subset & \dots
 \end{array}$$

The standard invariant **is** a subfactor planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  where

$$A_{k,+} = M'_0 \cap M_k, \quad A_{k,-} = M'_1 \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k-1,+}$$

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k+1,+}$$

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,+} \rightarrow A_{k-1,-}$$

$$P \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} : A_{k,-} \rightarrow A_{k+1,+}$$

# Relation to subfactors

Recall the standard invariant of a subfactor  $M_0 \subset M_1$ :

$$\begin{array}{ccccccc}
 \mathbb{C} & = & A_{0,+} & \subset & A_{1,+} & \subset & A_{2,+} & \subset & \dots \\
 & & & & \downarrow & & \downarrow & & \dots \\
 & & \mathbb{C} & = & A_{0,-} & \subset & A_{1,-} & \subset & \dots
 \end{array}$$

The standard invariant **is** a subfactor planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  where

$$A_{k,+} = M'_0 \cap M_k, \quad A_{k,-} = M'_1 \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

$$P \quad \langle \text{diagram} \rangle : A_{k,+} \rightarrow A_{k-1,+}$$

$$P \quad \langle \text{diagram} \rangle : A_{k,+} \rightarrow A_{k+1,+}$$

$$P \quad \langle \text{diagram} \rangle : A_{k,+} \rightarrow A_{k-1,-}$$

$$P \quad \langle \text{diagram} \rangle : A_{k,-} \rightarrow A_{k+1,+}$$

# Relation to subfactors

Recall the standard invariant of a subfactor  $M_0 \subset M_1$ :

$$\begin{array}{ccccccc}
 \mathbb{C} & = & A_{0,+} & \subset & A_{1,+} & \subset & A_{2,+} & \subset & \dots \\
 & & & & \uparrow & & \uparrow & & \dots \\
 & & \mathbb{C} & = & A_{0,-} & \subset & A_{1,-} & \subset & \dots
 \end{array}$$

The standard invariant **is** a subfactor planar algebra  $(A_{n,\pm})_{n \in \mathbb{N}_0}$  where

$$A_{k,+} = M'_0 \cap M_k, \quad A_{k,-} = M'_1 \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

$$P \quad \text{[Diagram]} : A_{k,+} \rightarrow A_{k-1,+}$$

$$P \quad \text{[Diagram]} : A_{k,+} \rightarrow A_{k+1,+}$$

$$P \quad \text{[Diagram]} : A_{k,+} \rightarrow A_{k-1,-}$$

$$P \quad \text{[Diagram]} : A_{k,-} \rightarrow A_{k+1,+}$$



# A subfactor planar algebra hierarchy

A  **$k$ -generated planar algebra** is a subfactor planar algebra generated by

$$A_{0,+} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \end{array} \right\}, \quad A_{0,-} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowright \end{array} \right\},$$

$$A_{1,+} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \\ | \end{array} \right\}, \quad A_{1,-} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowright \\ | \end{array} \right\},$$

$$A_{2,+} = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \end{array} \right\}, \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \\ \text{ ) ( } \end{array}, \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \\ \text{ ) ( } \\ \text{ ) ( } \end{array}, \dots, \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \\ \text{ ) ( } \\ \text{ ) ( } \\ \text{ ) ( } \end{array} \right\}$$

together with the ‘what you see is what you get’ action of planar tangles.  
For example,  $A_{2,-}$  is generated by the following rotation tangle:

$$S = \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \end{array}, \quad \text{e.g.} \quad P_S \left( \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \end{array} \right) = \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \\ \text{ ) ( } \end{array} = \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{ ) ( } \end{array}$$

# A subfactor planar algebra hierarchy

**0-generated:** Temperley–Lieb (TL)

Generators:  $\{ \text{cup}, \text{cap} \}$

Action:  $P_T(\text{cap}, \text{cup}) = \text{TL diagram} = \delta \cdot \text{cup}$

**1-generated:** e.g. Fuss–Catalan (FC)

Generators:  $\{ \text{cup}, \text{cap}, \text{triple} \}$

Action:  $P_T(\text{cap}, \text{triple}) = \text{FC diagram} = \gamma_1 \cdot \text{triple}$

# A subfactor planar algebra hierarchy

**1-generated:** e.g. Birman–Wenzl–Murakami (BMW)

Generators:  $\{ \textcircled{\cdot} \textcircled{\cdot}, \textcircled{\cdot} \textcircled{\cdot}, \textcircled{\cdot} \textcircled{\cdot} \}$

Relations:  $\textcircled{\cdot} \textcircled{\cdot} - \textcircled{\cdot} \textcircled{\cdot} = Q[\textcircled{\cdot} \textcircled{\cdot} - \textcircled{\cdot} \textcircled{\cdot}], \quad \textcircled{\cdot} \textcircled{\cdot} = \tau \textcircled{\cdot} \textcircled{\cdot}$

Action:  $P_T(\textcircled{\cdot} \textcircled{\cdot}, \textcircled{\cdot} \textcircled{\cdot}) = \textcircled{\cdot} \textcircled{\cdot} = \tau \textcircled{\cdot} \textcircled{\cdot}$

# A subfactor planar algebra hierarchy

1-generated: e.g. Liu

Generators:  $\{ \text{cup}, \text{cap}, \text{cross} \}$

Relations:  $\text{cup} = \text{cup} - \frac{1}{\delta} \text{cap}$ ,  $\text{cup} = 0$ ,  $\text{cross} = \epsilon \text{cross}$ ,  
+ braid-like relation

Action:  $P_T(\text{cap}, \text{cross}) = \text{diagram} = 0$

# Quantum integrability

# Classical integrability

A **classical system**  $H(\mathbf{p}, \mathbf{q})$  on a  $2n$ -dim. phase space is **integrable** if:

$$\{Q_i, Q_j\} = \{Q_i, H\} = 0, \quad \forall Q_i, Q_j \in \mathbf{Q} := \{Q_1, \dots, Q_n\}$$

and there are no functional relations among the **integrals of motion**  $Q_i$ .

**Classical integrable** systems are:

- Solvable – equations of motion can be determined explicitly
- Non-ergodic – dynamics are constrained to subspace of phase space

# Quantum integrability

Naïve attempt:  $\{\cdot, \cdot\} \mapsto \frac{i}{\hbar}[\cdot, \cdot]$

A **quantum system**  $H$  acting on a  $n$ -dim. Hilbert space is **integrable** if:

$$[Q_i, Q_j] = [Q_i, H] = 0, \quad \forall Q_i, Q_j \in \mathbf{Q} := \{Q_1, \dots, Q_n\}$$

and the **integrals of motion**  $Q_i$  are linearly independent.

Under this definition, all quantum systems are integrable! Diagonalise  $H$ :

$$H = \sum_{i=1}^n \lambda_i P_i, \quad [P_i, P_j] = [P_i, H] = 0, \quad \forall i, j = 1, \dots, n.$$

We seek a definition where such models are **solvable** and **non-ergodic**.

**Idea:** Examine the structure of the integrals of motion (IOM).

# Quantum integrability à la Caux and Mossel

To the increasing sequence of integers  $(N_1, N_2, N_3, \dots)$  associate **tower**:

$$\text{Hilbert spaces: } (\mathcal{H}^{(N_1)}, \mathcal{H}^{(N_2)}, \mathcal{H}^{(N_3)}, \dots)$$

where  $\mathcal{H}^{(N_a)} \subset \mathcal{H}^{(N_{a+1})}$  and  $\mathcal{A}$  is an algorithm that acts as:

$$\mathcal{H}^{(N_a)} \mapsto H^{(N_a)}, \quad \mathcal{H}^{(N_a)} \mapsto \mathbf{Q}^{(N_a)}$$

A **quantum system**  $H$  acting on a Hilbert space  $\mathcal{H}$  is **integrable** if it can be embedded within a **tower** such that:

- The number of IOM becomes unbounded i.e.  $\lim_{a \rightarrow \infty} |\mathbf{Q}^{(N_a)}| \rightarrow \infty$
- Each  $Q_i^{(N_a)} \in \mathbf{Q}^{(N_a)}$  can be embedded into  $(Q_i^{(N_1)}, Q_i^{(N_2)}, Q_i^{(N_3)}, \dots)$  and the number of nonzero matrix elements grows sub-exponentially

**Idea:** Observables are ergodic in systems with exp. growth. Exclude these.



# Planar-algebraic models

# Transfer operators

## Transfer tangles:

$$T_{n,+} = \text{Diagram 1}, \quad T_{n,-} = \text{Diagram 2}$$

**R-operator:** Let  $B_{n,\pm}$  denote a basis for  $A_{n,\pm}$  and  $u \in \Omega \subseteq \mathbb{C}$

$$R_{\pm}(u) = \text{Diagram 3} = \sum_{a \in B_{2,\pm}} r_a(u) a, \quad A_{2,\pm} = \langle R_{\pm}(u) \mid u \in \Omega \rangle_{\mathbb{P}}$$

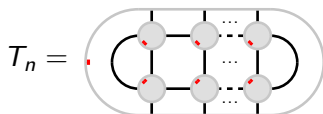
## Homogeneous transfer operators:

$$T_{n,+}(u) = \text{Diagram 4}, \quad T_{n,-}(u) = \text{Diagram 5}$$

Implies that the underlying planar algebra is **unshaded**. Ignore shading!

# Transfer operators

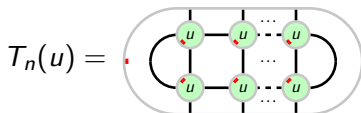
## Transfer tangles:



**R-operator:** Let  $B_n$  denote a basis for  $A_n$  and  $u \in \Omega \subseteq \mathbb{C}$

$$R(u) = \text{⊗}_{u} = \sum_{a \in B_2} r_a(u) a, \quad A_2 = \langle R(u) \mid u \in \Omega \rangle_{\mathcal{P}}$$

## Homogeneous transfer operators:



Implies that the underlying planar algebra is **unshaded**. Ignore shading!

# Planar-algebraic models

Let  $(A_n)_{n \in \mathbb{N}_0}$  denote an (unshaded) subfactor planar algebra. Consider a system described by a **transfer operator**  $T_n(u) \in A_n$  satisfying

$$[T_n(u), T_n(v)] = 0, \quad \forall u, v \in \Omega \subseteq \mathbb{C}.$$

The corresponding **tower** with  $(N_1, N_2, N_3, \dots) = (2, 4, 6, \dots)$  is given by:

**Hilbert spaces:**  $(A_2, A_4, A_6, \dots)$

where the algorithm  $\mathcal{A}$  assigns hamiltonians and corresponding IOM as

$$T_n(u) = \sum_{i=0}^{\infty} u^i Q_n^{(i)}$$

where  $H_n \equiv Q_{2n}^{(1)}$  and  $\mathbf{Q}_n := \{Q_{2n}^{(i)} \mid i = 2, 3, \dots\}$ .

# Yang–Baxter integrability

A model is **Yang–Baxter integrable** if the  $R$ -operator satisfies a set of **local relations** that imply  $[T_n(u), T_n(v)] = 0$ .

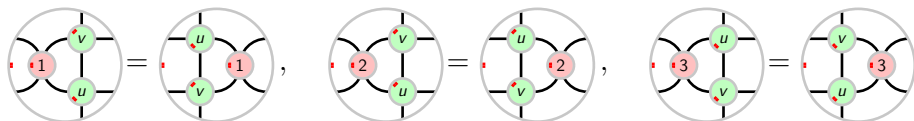
For the **homogeneous transfer operator** a set is given by:

- Inversion identities



$$= \quad (i = 1, 2, 3)$$

- Yang–Baxter equations



+ boundary Yang–Baxter equations

Such a model is called **homogeneous Yang–Baxter integrable (HYB)**.

# Yang–Baxter relation planar algebras

An (unshaded) subfactor planar algebra is **Yang–Baxter relation** (YBR) if each triple  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A_2$  satisfy

$$\text{Diagram 1} = \sum_{\mathbf{a}, \mathbf{b}, \mathbf{c} \in B_2} C_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}} \text{Diagram 2}, \quad C_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}} \in \mathbb{C}$$

YBR planar algebras naturally give rise to **quantum integrable models**

**Quantum:** Subfactor property – each  $A_n$  are Hilbert spaces

**Integrable:** YBR property – natural YB structure on  $A_2$  and  $A_3$

The 0-generated (TL) planar algebra is YBR and admits a HYB model.

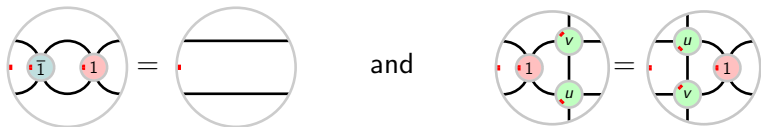
# Integrable models

## Theorem (XP, Rasmussen '23)

A 1-generated planar algebra admits a **homogeneous Yang–Baxter integrable model** if and only if it is **Yang–Baxter relation**.

**Sketch:** HYB  $\Rightarrow$  YBR

With the **proto-1-generated planar algebra**, no non-trivial solution to



unless a YBR is imposed on the planar algebra.

# Integrable models

## Theorem (XP, Rasmussen '23)

A 1-generated planar algebra admits a **homogeneous Yang–Baxter integrable model** if and only if it is **Yang–Baxter relation**.

**Sketch:** HYB  $\Leftarrow$  YBR

## Theorem (Liu '15)

A 1-generated YBR planar algebra is isomorphic to a **Fuss–Catalan (FC)**, **Birman–Wenzl–Murakami (BMW)** or **Liu** planar algebra.

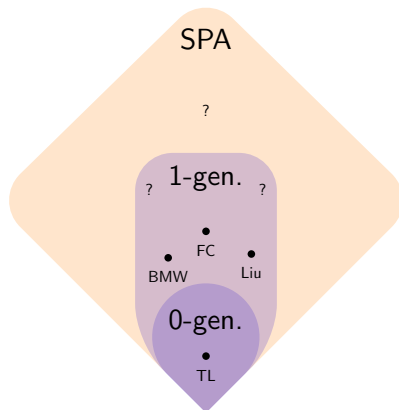
$$\text{FC: } \begin{array}{c} \text{X} \\ \text{u} \\ \text{X} \end{array} = r_{\mathbb{1}}(u) \begin{array}{c} \text{X} \\ \text{O} \\ \text{X} \end{array} + r_E(u) \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \end{array} + r_P(u) \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \end{array} \quad (\text{Di Francesco '98})$$

$$\text{BMW: } \begin{array}{c} \text{X} \\ \text{u} \\ \text{X} \end{array} = r_{\mathbb{1}}(u) \begin{array}{c} \text{X} \\ \text{O} \\ \text{X} \end{array} + r_e(u) \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \end{array} + r_g(u) \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \end{array} \quad (\text{Cheng, Ge, Xue '91})$$

$$\text{Liu: } \begin{array}{c} \text{X} \\ \text{u} \\ \text{X} \end{array} = r_{\mathbb{1}}(u) \begin{array}{c} \text{X} \\ \text{O} \\ \text{X} \end{array} + r_e(u) \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \end{array} + r_s(u) \begin{array}{c} \text{X} \\ \text{X} \\ \text{X} \end{array} \quad (\text{XP, Rasmussen '23})$$



## The story so far...

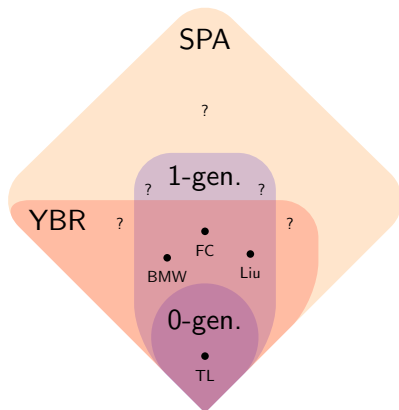


SPA – Subfactor planar algebras

YBR – Yang–Baxter relation planar algebras

HYB – Homogeneous Yang–Baxter integrable models

## The story so far...

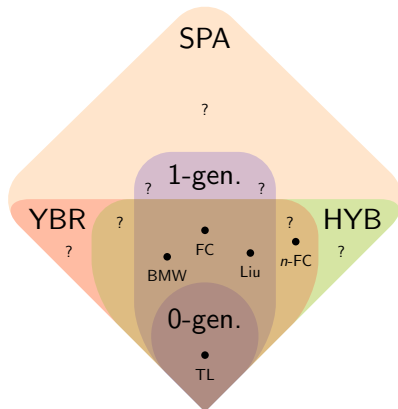


SPA – Subfactor planar algebras

YBR – Yang–Baxter relation planar algebras

HYB – Homogeneous Yang–Baxter integrable models

## The story so far...



SPA – Subfactor planar algebras

YBR – Yang–Baxter relation planar algebras

HYB – Homogeneous Yang–Baxter integrable models

# Outlook

# Outlook

## Summary:

- Subfactors encode quantum integrable models
- Relevant 1-generated planar algebras are necessarily YBR
- ‘quantum’  $\leftrightarrow$  ‘subfactor’ and ‘integrable’  $\leftrightarrow$  ‘YBR’
- 1-generated planar algebras are just the beginning!

## Future work:

- Extend results to 2-generated planar algebras
- Consider models described by an inhomogenous transfer operator
- Cylindrical and annular models
- Continuum scaling limit – subfactors and conformal field theories?

The end!