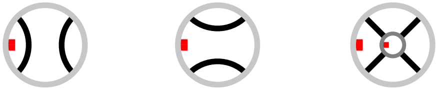
Subfactors and quantum integrability

Xavier Poncini

University of Queensland



University of Melbourne Mathematical Physics Seminar

Based on work (arXiv:2206.14462 + arXiv:2302.11712) with Jørgen Rasmussen

Motivation

Subfactors appear in many interacting areas of mathematical physics:

- tensor categories
- quantum groups
- vertex operators algebras
- quantum field theory
- knot theory
- statistical mechanics – **quantum integrability**

Question

Which **subfactors** encode the structure of **quantum integrable models**?

We will encounter subfactors in their incarnation as **planar algebras**.

Outline

Subfactors

Subfactors

A **factor** is an infinite-dimensional von Neumann algebra with **trivial centre** and a **trace**. For two factors $M_0 \subset M_1$, M_0 is a **subfactor** of M_1.

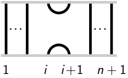
Further subfactors can be produced via **Jones' basic construction**:

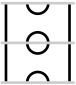
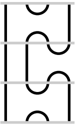
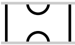
$$M_0 \subset M_1 \subset^{e_1} M_2 \subset^{e_2} M_3 \subset \dots \qquad \text{where} \;\; M_{k+1} := \langle M_k, e_k \rangle$$

and $\langle e_1, e_2, \dots, e_n \rangle$ is a finite-dim. C^*-algebra subject to the relations:

$$e_i^2 = e_i = e_i^*, \qquad e_i e_{i\pm1} e_i = \delta^{-2} e_i, \qquad e_i e_j = e_j e_i, \qquad |i-j| \geq 2$$

This is the **Temperley–Lieb** (TL) **algebra** which can be expressed:

$e_i \leftrightarrow \frac{1}{\delta}$ 

, $e.g$  $= \delta$  ,  $=$ 

The standard invariant

The **standard invariant** of a subfactor consists of the two **towers**:

$$\begin{array}{ccccccccc} Z(M_0) & = & M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & \cdots \\ & & & & \cup & & \cup & & \\ & & Z(M_1) & = & M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & \cdots \end{array}$$

where

$$M_n' \cap M_m = \{x \in M_m \,|\, xy = yx, \forall\, y \in M_n\}, \qquad n \in \{0, 1\}$$

are finite dimensional C^*-algebras that include Temperley-Lieb algebras.

Thanks to a theorem of Popa, a subfactor can be **reconstructed** from the standard invariant. The standard invariant **stores** the data of a subfactor.

Planar algebras provide a pictorial description of the standard invariant.

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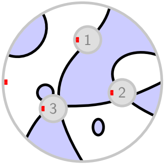
Planar algebras provide a pictorial description of the standard invariant.

Planar algebras

Definition

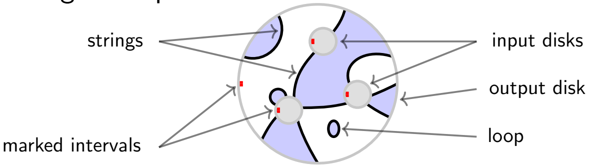
A **planar algebra** is a collection of vector spaces $(A_{n,\pm})_{n\in\mathbb{N}_0}$, together with the action of **shaded planar tangles** as multilinear maps e.g.

$T =$ 

$,\qquad \mathrm{P}_T : A_{1,+} \times A_{2,-} \times A_{3,+} \to A_{4,+}$

such that this action is compatible with the composition of tangles.

Shaded planar tangle components:



Planar algebras naturally describe interactions in two-dimensions!

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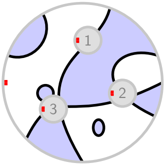
$T =$ 

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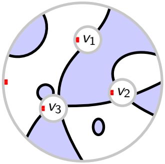
Multilinear map action:

$\mathrm{P}_T(v_1, v_2, v_3) =$ 

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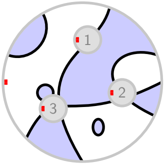
$T =$ 

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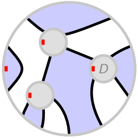
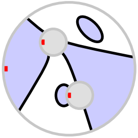
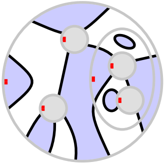
Composition of tangles:

$T =$ 

, $S =$ 
, $T \circ_D S =$ 

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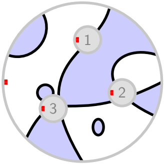
$T =$ 

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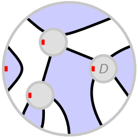
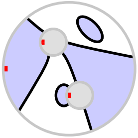
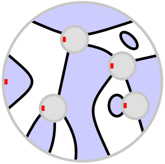
$T =$ 

, $S =$ 
, $T \circ_D S =$ 

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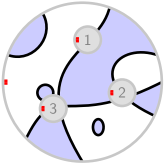
$T =$ 

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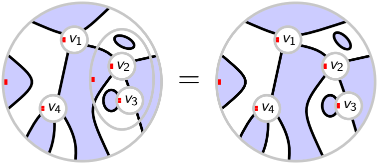
Compatibility condition:

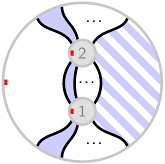


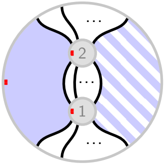
$\in A_{5,-}$

Planar algebras naturally describe interactions in two-dimensions!

Planar algebras

A planar algebra contains countably many **associative algebras**:

$M_{n,+} :=$ 

, $\qquad M_{n,-} :=$ 

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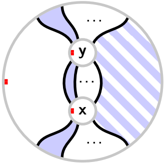
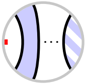
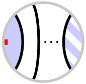
with $M_{n,\pm}$ inducing a multiplication on $A_{n,\pm}$ e.g.

$\mathbf{xy} =$ 

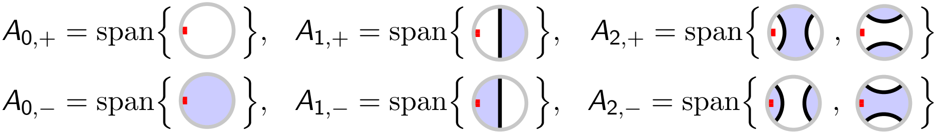
$= \mathsf{P}_{M_{n,+}}(\mathbf{x}, \mathbf{y}), \qquad \forall\, \mathbf{x}, \mathbf{y} \in A_{n,+}.$

Under mild conditions the algebras are **unital** with units

$\mathbb{1}_{n,\pm} := \mathsf{P}_{\mathrm{Id}_{n,\pm}}(), \qquad \mathrm{Id}_{n,+} :=$  , $\qquad \mathrm{Id}_{n,-} :=$  .

Example: Temperley-Lieb planar algebra

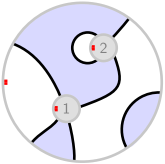
Vector spaces:

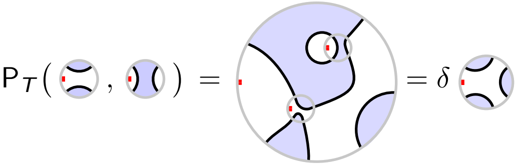


$$\dots$$

Planar tangle action:

$$\mathrm{P}_T : A_{2,+} \times A_{2,+} \to A_{3,+}$$

$T =$ 



Every planar algebra contains Temperley–Lieb-like planar algebra!

Example: Temperley-Lieb planar algebra

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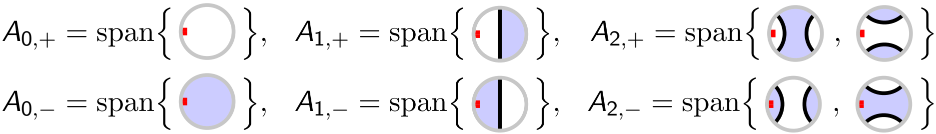

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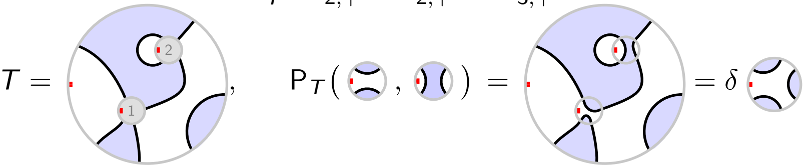

The shading of a planar algebra need not carry any non-trivial information. In this case, it can be ignored and the planar algebra is called **unshaded**.

Example: Temperley-Lieb planar algebra

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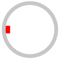
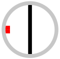
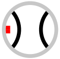
$A_0 = \mathrm{span}\{$  $\}$, $A_1 = \mathrm{span}\{$  $\}$, $A_2 = \mathrm{span}\{$  ,  $\}$,

$$\dots$$

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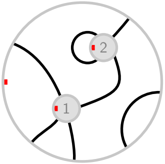
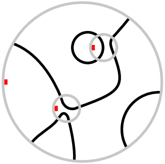
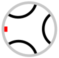
$$\mathrm{P}_T \colon A_2 \times A_2 \to A_3$$

$T =$ 

, $\mathrm{P}_T($  ,  $) =$  $= \delta$ 

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Example: Tensor planar algebra

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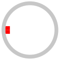
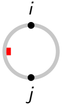
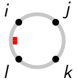
$A_{0,\pm} = \mathrm{span}\Big\{$  $\Big\}$, $\qquad A_{1,\pm} = \mathrm{span}\Big\{$ 

$\Big|\, i,j = 1,\ldots,N\Big\}$,

$A_{2,\pm} = \mathrm{span}\Big\{$ 

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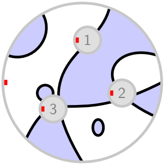
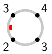
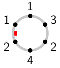
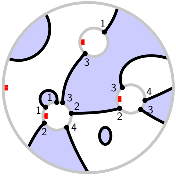
$T =$ 

, $\quad \mathrm{P}_T\Big($ 

, 

, 

$\Big) =$ 

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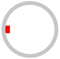
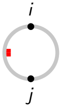
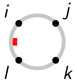
$A_{0,\pm} = \mathrm{span}\Big\{$  $\Big\}$, $\quad A_{1,\pm} = \mathrm{span}\Big\{$  $\Big| \, i,j = 1,\dots,N \Big\}$,

$A_{2,\pm} = \mathrm{span}\Big\{$  $\Big| \, i,j,k,l = 1,\dots,N \Big\}$, $\qquad \dots$

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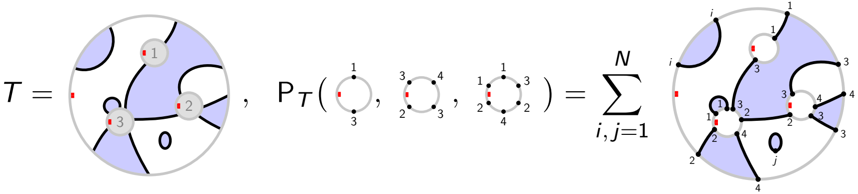

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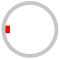
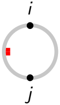
$A_{0,\pm} = \mathrm{span}\{$  $\}$, $A_{1,\pm} = \mathrm{span}\{$ 

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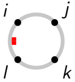
$A_{2,\pm} = \mathrm{span}\{$ 

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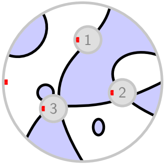
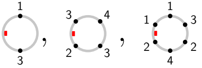
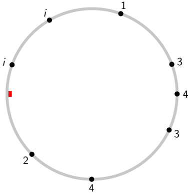
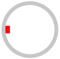
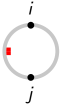
$T =$ 

, $\mathrm{P}_T\big($ 

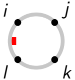
$\big) = \sum_{i,j=1}^{N}$ 

Example: Tensor planar algebra

Vector spaces:

$A_{0,\pm} = \text{span}\Big\{$  $\Big\}$, $A_{1,\pm} = \text{span}\Big\{$ 

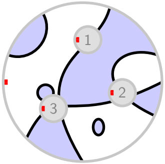
$\Big| \, i, j = 1, \dots, N \Big\}$,

$A_{2,\pm} = \text{span}\Big\{$ 

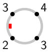
$\Big| \, i, j, k, l = 1, \dots, N \Big\}$, $\dots$

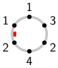
Planar tangle action:

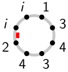
$$\mathsf{P}_T : A_{1,+} \times A_{2,-} \times A_{3,+} \to A_{4,+}$$

$T =$ 

, $\mathsf{P}_T\Big($ 

, 

, 

$\Big) = N \sum_{i=1}^{N}$ 

The shading of a planar algebra need not carry any non-trivial information. In this case, it can be ignored and the planar algebra is called **unshaded**.

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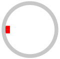
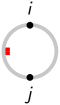
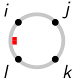
$A_0 = \mathrm{span}\Big\{$  $\Big\}$, $\quad A_1 = \mathrm{span}\Big\{$ 

$\Big| \, i, j = 1, \ldots, N \Big\}$,

$A_2 = \mathrm{span}\Big\{$ 

$\Big| \, i, j, k, l = 1, \ldots, N \Big\}$, $\quad \ldots$

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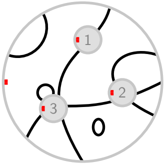
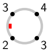
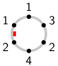
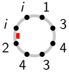
$T =$ 

, $\quad \mathrm{P}_T\Big($ 

, 

, 

$\Big) = N \sum_{i=1}^{N}$ 

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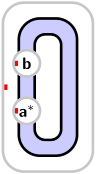
Subfactor planar algebras

Definition

A planar algebra $(A_{n,\pm})_{n\in\mathbb{N}_0}$ that is involutive, evaluable, spherical and positive-definite is called a **subfactor planar algebra**.

There is an inner product on each $A_{n,\pm}$, for example:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{2,+} :=$$



With multiplication tangle, each $A_{n,\pm}$ is a finite-dimensional C^*-algebra with a Temperley-Lieb subalgebra.

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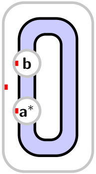
$\langle \mathbf{a}, \mathbf{b} \rangle_{2,+} :=$ 

, $\mathbf{a}^*$ well-defined

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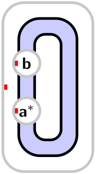

$$\in \mathbb{C}, \qquad \dim(A_{n,\pm}) < \infty$$

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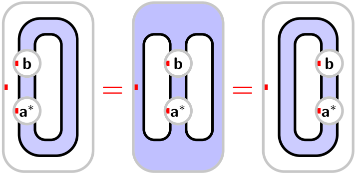

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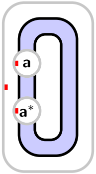
$\langle \mathbf{a}, \mathbf{a} \rangle_{2,+} :=$ 

> 0

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Relation to subfactors

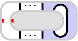
Recall the standard invariant of a subfactor $M_0 \subset M_1$:

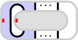
$$\begin{array}{ccccccccc} \mathbb{C} & = & M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & \cdots \\ & & & & \cup & & \cup & & \cdots \\ & & \mathbb{C} & = & M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & \cdots \end{array}$$

The standard invariant **is** a subfactor planar algebra $(A_{n,\pm})_{n\in\mathbb{N}_0}$ where

$$A_{k,+} = M_0' \cap M_k, \qquad A_{k,-} = M_1' \cap M_{k+1}$$

The subfactor planar algebra **stores** the data of a subfactor. We note e.g:

P  $: A_{k,+} \to A_{k-1,+},$ P  $: A_{k,+} \to A_{k+1,+},$

P  $: A_{k,+} \to A_{k-1,-},$ P  $: A_{k,-} \to A_{k+1,+}$

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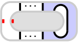
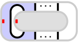
P  $: A_{k,+} \to A_{k-1,+},$ P  $: A_{k,+} \to A_{k+1,+},$

P  $: A_{k,+} \to A_{k-1,-},$ P  $: A_{k,-} \to A_{k+1,+}$

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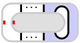
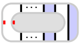
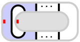
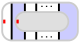
P  $: A_{k,+} \to A_{k-1,+}$,

P  $: A_{k,+} \to A_{k+1,+}$,

P  $: A_{k,+} \to A_{k-1,-}$,

P  $: A_{k,-} \to A_{k+1,+}$

Relation to subfactors

Recall the standard invariant of a subfactor $M_0 \subset M_1$:

$$\begin{array}{ccccccccccc} \mathbb{C} & = & A_{0,+} & \color{red}{\to} & A_{1,+} & \color{red}{\to} & A_{2,+} & \color{red}{\to} & \cdots \\ & & & & \cup & & \cup & & \cdots \\ & & \mathbb{C} & = & A_{0,-} & \subset & A_{1,-} & \subset & \cdots \end{array}$$

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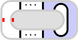
P  : $A_{k,+} \to A_{k-1,+}$, P  : $\color{red}{A_{k,+} \to A_{k+1,+}}$,

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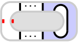
P  $: A_{k,+} \to A_{k-1,+}$,

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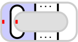
P  $: A_{k,+} \to A_{k-1,-}$,

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\mathbb{C} & = & A_{0,+} & \subset & A_{1,+} & \subset & A_{2,+} & \subset & \cdots \\
 & & & & {\color{red}\uparrow} & & {\color{red}\uparrow} & & \cdots \\
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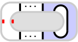
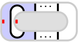
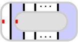
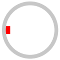
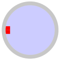
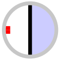
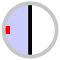
P  $: A_{k,+} \to A_{k-1,+},$ P  $: A_{k,+} \to A_{k+1,+},$

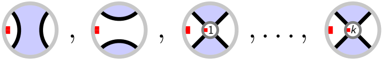
P  $: A_{k,+} \to A_{k-1,-},$ P  $: A_{k,-} \to A_{k+1,+}$

A subfactor planar algebra hierarchy

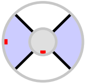
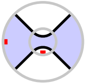
A k-**generated planar algebra** is a subfactor planar algebra generated by

$A_{0,+} = \mathrm{span}\{$  $\}$, $A_{0,-} = \mathrm{span}\{$  $\}$,

$A_{1,+} = \mathrm{span}\{$  $\}$, $A_{1,-} = \mathrm{span}\{$  $\}$,

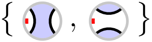
$A_{2,+} = \mathrm{span}\{$  $\}$

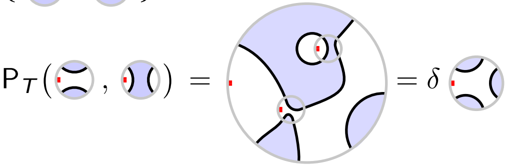
together with the 'what you see is what you get' action of planar tangles. For example, $A_{2,-}$ is generated by the following rotation tangle:

$S =$  , e.g. $\mathrm{P}_S($  $) =$  $=$ 

A subfactor planar algebra hierarchy

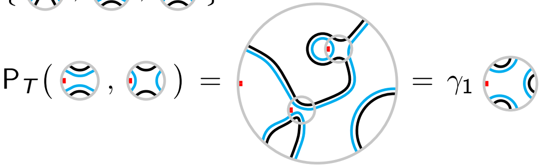
0-**generated:** Temperley–Lieb (TL)

Generators: 

Action: 

1-**generated:** e.g. Fuss–Catalan (FC)

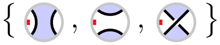
Generators: 

Action: 

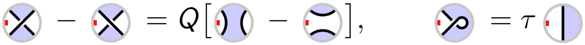
A subfactor planar algebra hierarchy

1-**generated:** e.g. Birman–Wenzl–Murakami (BMW)

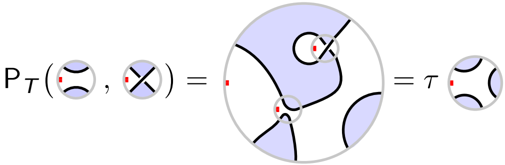
Generators:



Relations:



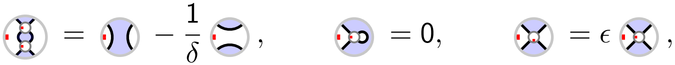
Action:



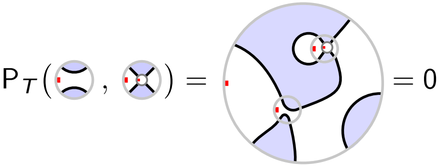
A subfactor planar algebra hierarchy

1-**generated:** e.g. Liu

Generators: 

Relations: 

+ braid-like relation

Action: 

Quantum integrability

Classical integrability

A **classical system** $H(\mathbf{p}, \mathbf{q})$ on a $2n$-dim. phase space is **integrable** if:

$$\{Q_i, Q_j\} = \{Q_i, H\} = 0, \qquad \forall\ Q_i, Q_j \in \mathbf{Q} := \{Q_1, \dots, Q_n\}$$

and there are no functional relations among the **integrals of motion** Q_i.

Classical integrable systems are:

- Solvable – equations of motion can be determined explicitly
- Non-ergodic – dynamics are constrained to subspace of phase space

Quantum integrability

Naïve attempt: $\{\cdot,\cdot\} \mapsto \frac{\mathrm{i}}{\hbar}[\cdot,\cdot]$

A **quantum system** H acting on a n-dim. Hilbert space is **integrable** if:

$$[Q_i, Q_j] = [Q_i, H] = 0, \qquad \forall\, Q_i, Q_j \in \mathbf{Q} := \{Q_1, \dots, Q_n\}$$

and the **integrals of motion** Q_i are linearly independent.

Under this definition, all quantum systems are integrable! Diagonalise H:

$$H = \sum_{i=1}^{n} \lambda_i P_i, \qquad [P_i, P_j] = [P_i, H] = 0, \qquad \forall\, i, j = 1, \dots n.$$

We seek a definition where such models are **solvable** and **non-ergodic**.

Idea: Examine the structure of the integrals of motion (IOM).

Quantum integrability à la Caux and Mossel

To the increasing sequence of integers $(N_1, N_2, N_3, \ldots)$ associate **tower**:

$$\textbf{Hilbert spaces:} \qquad (\mathcal{H}^{(N_1)}, \mathcal{H}^{(N_2)}, \mathcal{H}^{(N_3)}, \ldots)$$

where $\mathcal{H}^{(N_a)} \subset \mathcal{H}^{(N_{a+1})}$ and $\mathcal{A}$ is an algorithm that acts as:

$$\mathcal{H}^{(N_a)} \mapsto H^{(N_a)}, \qquad \mathcal{H}^{(N_a)} \mapsto \mathbf{Q}^{(N_a)}$$

A **quantum system** H acting on a Hilbert space $\mathcal{H}$ is **integrable** if it can be embedded within a **tower** such that:

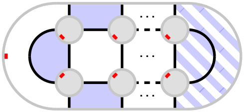
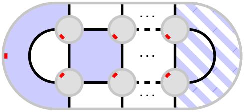
- The number of IOM becomes unbounded i.e. $\lim_{a\to\infty} |\mathbf{Q}^{(N_a)}| \to \infty$
- Each $Q_i^{(N_a)} \in \mathbf{Q}^{(N_a)}$ can be embedded into $(Q_i^{(N_1)}, Q_i^{(N_2)}, Q_i^{(N_3)}, \ldots)$ and the number of nonzero matrix elements grows sub-exponentially

Idea: Observables are ergodic in systems with exp. growth. Exclude these.

Planar-algebraic models

Transfer operators

Transfer tangles:

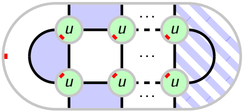
$T_{n,+} =$  , $T_{n,-} =$ 

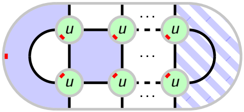
***R*-operator:** Let $B_{n,\pm}$ denote a basis for $A_{n,\pm}$ and $u \in \Omega \subseteq \mathbb{C}$

$R_\pm(u) =$ 

$= \sum_{a \in B_{2,\pm}} r_a(u)\, a, \qquad A_{2,\pm} = \langle R_\pm(u) \,|\, u \in \Omega \rangle_{\mathrm{P}}$

Homogeneous transfer operators:

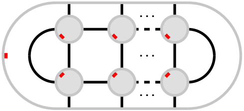
$T_{n,+}(u) =$ 

, $T_{n,-}(u) =$ 

Implies that the underlying planar algebra is **unshaded**. Ignore shading!

Transfer operators

Transfer tangles:

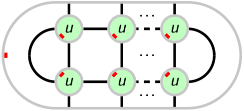
$T_n =$ 

R-**operator:** Let B_n denote a basis for A_n and $u \in \Omega \subseteq \mathbb{C}$

$R(u) =$ 

$= \sum_{a \in B_2} r_a(u)\, a, \qquad A_2 = \langle R(u) \mid u \in \Omega \rangle_{\mathsf{P}}$

Homogeneous transfer operators:

$T_n(u) =$ 

Implies that the underlying planar algebra is **unshaded**. Ignore shading!

Planar-algebraic models

Let $(A_n)_{n\in\mathbb{N}_0}$ denote an (unshaded) subfactor planar algebra. Consider a system described by a **transfer operator** $T_n(u) \in A_n$ satisfying

$$[T_n(u), T_n(v)] = 0, \qquad \forall\, u, v \in \Omega \subseteq \mathbb{C}.$$

The corresponding **tower** with $(N_1, N_2, N_3, \ldots) = (2, 4, 6, \ldots)$ is given by:

Hilbert spaces: $(A_2, A_4, A_6, \ldots)$

where the algorithm $\mathcal{A}$ assigns hamiltonians and corresponding IOM as

$$T_n(u) = \sum_{i=0}^{\infty} u^i\, Q_n^{(i)}$$

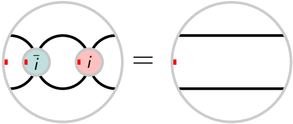
where $H_n \equiv Q_{2n}^{(1)}$ and $\mathbf{Q}_n := \{Q_{2n}^{(i)} \mid i = 2, 3, \ldots\}$.

Yang–Baxter integrability

A model is **Yang–Baxter integrable** if the R-operator satisfies a set of **local relations** that imply $[T_n(u), T_n(v)] = 0$.

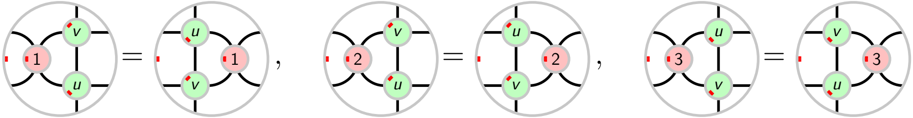
For the **homogeneous transfer operator** a set is given by:

- Inversion identities



$(i = 1, 2, 3)$

- Yang–Baxter equations

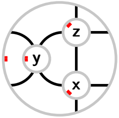


$+$ boundary Yang–Baxter equations

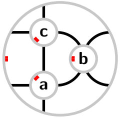
Such a model is called **homogeneous Yang–Baxter integrable** (HYB).

Yang–Baxter relation planar algebras

An (unshaded) subfactor planar algebra is **Yang–Baxter relation** (YBR) if each triple $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A_2$ satisfy



$$= \sum_{\mathbf{a},\mathbf{b},\mathbf{c} \in B_2} C^{\mathbf{a},\mathbf{b},\mathbf{c}}_{\mathbf{x},\mathbf{y},\mathbf{z}}$$



$$, \qquad C^{\mathbf{a},\mathbf{b},\mathbf{c}}_{\mathbf{x},\mathbf{y},\mathbf{z}} \in \mathbb{C}$$

YBR planar algebras naturally give rise to **quantum integrable models**

Quantum: Subfactor property – each A_n are Hilbert spaces

Integrable: YBR property – natural YB structure on A_2 and A_3

The 0-generated (TL) planar algebra is YBR and admits a HYB model.

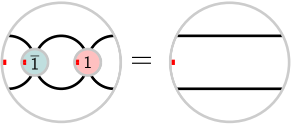
Integrable models

Theorem (XP, Rasmussen '23)

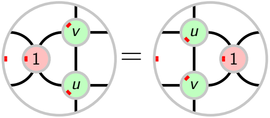
A 1-generated planar algebra admits a **homogeneous Yang–Baxter integrable model** if and only if it is **Yang–Baxter relation**.

Sketch: HYB ⇒ YBR

With the **proto-1-generated planar algebra**, no non-trivial solution to



and



unless a YBR is imposed on the planar algebra.

Integrable models

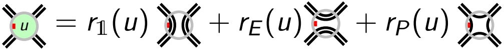
Theorem (XP, Rasmussen '23)

A 1-generated planar algebra admits a **homogeneous Yang–Baxter integrable model** if and only if it is **Yang–Baxter relation**.

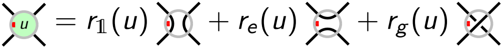
Sketch: HYB $\Leftarrow$ YBR

Theorem (Liu '15)

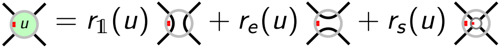
A 1-generated YBR planar algebra is isomorphic to a **Fuss–Catalan** (FC), **Birman–Wenzl–Murakami** (BMW) or **Liu** planar algebra.

FC: 

(Di Francesco '98)

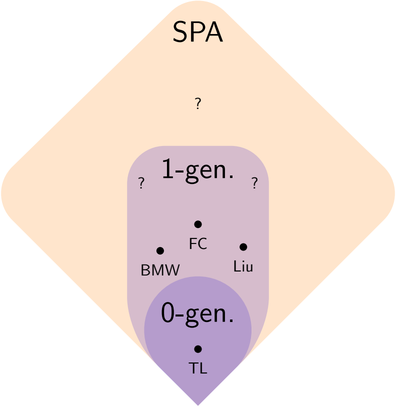
BMW: 

(Cheng, Ge, Xue '91)

Liu: 

(XP, Rasmussen '23)

The story so far...

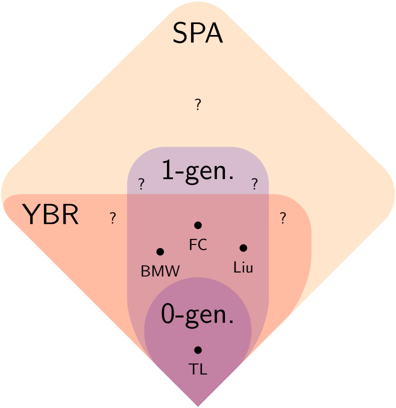


SPA – Subfactor planar algebras

YBR – Yang–Baxter relation planar algebras

HYB – Homogeneous Yang-Baxter integrable models

The story so far...



SPA – Subfactor planar algebras

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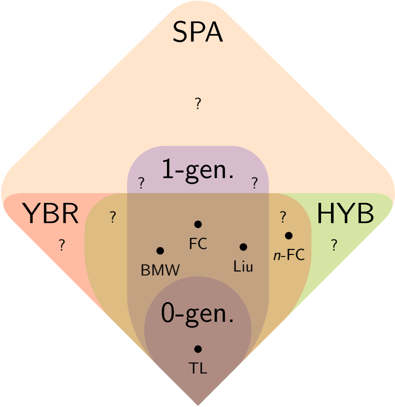

SPA – Subfactor planar algebras

YBR – Yang–Baxter relation planar algebras

HYB – Homogeneous Yang-Baxter integrable models

Outlook

Outlook

Summary:

- Subfactors encode quantum integrable models
- Relevant 1-generated planar algebras are necessarily YBR
- ‘quantum’ $\leftrightsquigarrow$ ‘subfactor’ and ‘integrable’ $\leftrightsquigarrow$ ‘YBR’
- 1-generated planar algebras are just the beginning!

Future work:

- Extend results to 2-generated planar algebras
- Consider models described by an inhomogenous transfer operator
- Cylindrical and annular models
- Continuum scaling limit – subfactors and conformal field theories?

The end!