A planar-algebraic universe

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Motivation

Some suggestions that spacetime is not a continuum:

- Aspects of the ‘It from Qubit’ program
- Successes of loop quantum gravity and causal sets
- Computational universe hypothesis

Question

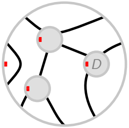
Can we discretise our theories such that they are recovered under a continuum limit?

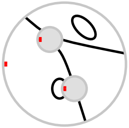
Outline

Planar algebras

Planar algebras

Planar tangles

$R =$ 

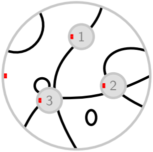
, $S =$ 

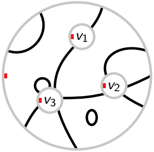
Graded vector space

$$(P_n)_{n\in\mathbb{N}_0}$$

Multilinear maps

$$\mathrm{P}_T \colon P_2 \times P_4 \times P_6 \to P_8$$

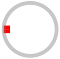
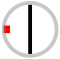
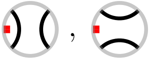
$T =$ 

, $\mathrm{P}_T(v_1, v_2, v_3) =$ 

$\in P_8$

Example: Temperley-Lieb planar algebra

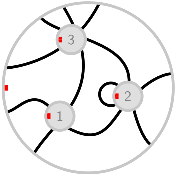
Vector spaces

$P_0 = \text{span}\{$  $\}$, $P_2 = \text{span}\{$  $\}$, $P_4 = \text{span}\{$  $\}$,

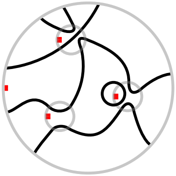
$P_6 = \text{span}\{$  $\}$, $\ldots$

Planar tangle action

$$\mathrm{P}_T : P_4 \times P_6 \times P_6 \to P_8$$

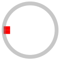
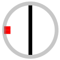
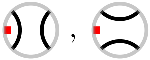
$T =$ 

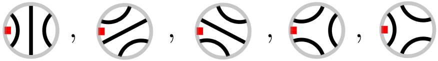
,

$\mathrm{P}_T($  $) =$ 

Example: Temperley-Lieb planar algebra

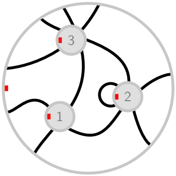
Vector spaces

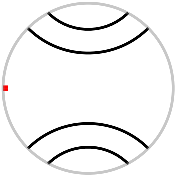
$P_0 = \mathrm{span}\{$  $\}$, $P_2 = \mathrm{span}\{$  $\}$, $P_4 = \mathrm{span}\{$  $\}$,

$P_6 = \mathrm{span}\{$  $\}$, $\dots$

Planar tangle action

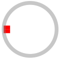
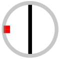
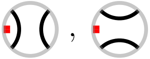
$$\mathrm{P}_T \colon P_4 \times P_6 \times P_6 \to P_8$$

$T =$ 

$\mathrm{P}_T($  $) = \delta$ 

Example: Temperley-Lieb planar algebra

Vector spaces

$P_0 = \mathrm{span}\{$  $\}$, $P_2 = \mathrm{span}\{$  $\}$, $P_4 = \mathrm{span}\{$  $\}$,

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Planar tangle action

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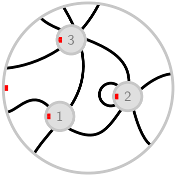
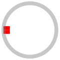
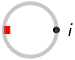
$T =$ 

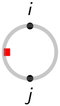
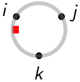
,

$\mathrm{P}_T($  $) = \delta$ 

Example: Tensor planar algebra

Vector spaces

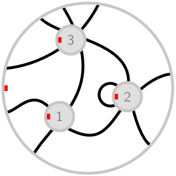
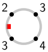
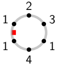
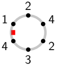
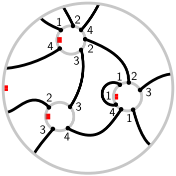
$P_0 = \mathrm{span}\Big\{$  $\Big\},\qquad P_1 = \mathrm{span}\Big\{$  $\Big|\, i = 1, \ldots, N\Big\},$

$P_2 = \mathrm{span}\Big\{$  $\Big|\, i, j = 1, \ldots, N\Big\},\quad P_3 = \mathrm{span}\Big\{$  $\Big|\, i, j, k = 1, \ldots, N\Big\},$

$\ldots$

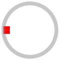
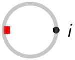
Planar tangle action

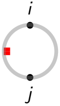
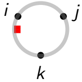
$$\mathsf{P}_T \colon P_4 \times P_6 \times P_6 \to P_8$$

$T =$  $,\quad \mathsf{P}_T\Big($  $,$  $,$  $\Big) =$ 

Example: Tensor planar algebra

Vector spaces

$P_0 = \mathrm{span}\{$  $\}$, $P_1 = \mathrm{span}\{$  $\mid i = 1, \ldots, N\}$,

$P_2 = \mathrm{span}\{$  $\mid i, j = 1, \ldots, N\}$, $P_3 = \mathrm{span}\{$  $\mid i, j, k = 1, \ldots, N\}$,

$\ldots$

Planar tangle action

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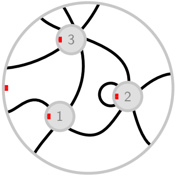
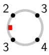
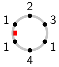
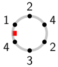
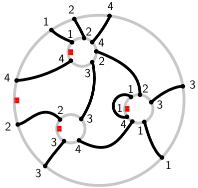
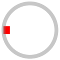
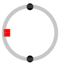
$T =$ , $\mathrm{P}_T($ , ,  $) =$ 

Example: Tensor planar algebra

Vector spaces

$P_0 = \mathrm{span}\Big\{$  $\Big\}$, $\quad P_1 = \mathrm{span}\Big\{$  $\Big| \ i = 1, \ldots, N \Big\}$,

$P_2 = \mathrm{span}\Big\{$  $\Big| \ i, j = 1, \ldots, N \Big\}$, $\quad P_3 = \mathrm{span}\Big\{$  $\Big| \ i, j, k = 1, \ldots, N \Big\}$,

$\ldots$

Planar tangle action

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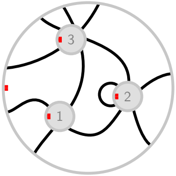
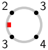
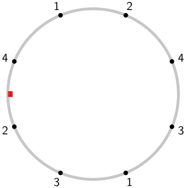
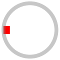
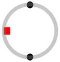
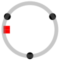
$T =$  , $\quad \mathrm{P}_T\Big($  ,  ,  $\Big) =$ 

Example: Tensor planar algebra

Vector spaces

$P_0 = \mathrm{span}\Big\{$  $\Big\}$, $P_1 = \mathrm{span}\Big\{$  $\Big|\, i = 1, \ldots, N \Big\}$,

$P_2 = \mathrm{span}\Big\{$  $\Big|\, i, j = 1, \ldots, N \Big\}$, $P_3 = \mathrm{span}\Big\{$  $\Big|\, i, j, k = 1, \ldots, N \Big\}$,

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Planar tangle action

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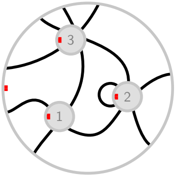
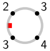
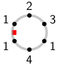
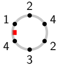
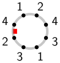
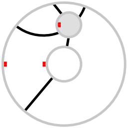
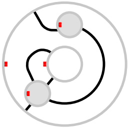
$T =$  ,

$\mathrm{P}_T\Big($  ,  ,  $\Big) =$ 

Annular tangles

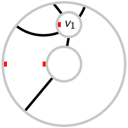
Annular tangles

$R =$  ,

$S =$ 

Graded annular vector space

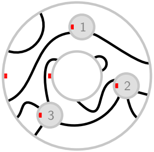
$$(V_{m,n})_{m,n\in\mathbb{N}_0}$$

e.g. 

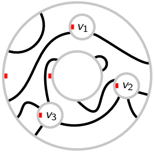
$\in V_{2,4}$

Multilinear maps

$$\mathrm{A}_T \colon P_2 \times P_4 \times P_4 \to V_{4,8}$$

$T =$ 

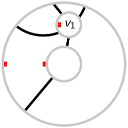
,

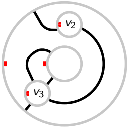
$\mathrm{A}_T(v_1, v_2, v_3) =$ 

$\in V_{4,8}$

Action of annular vectors

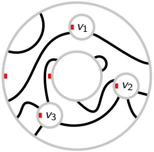
Annular vectors

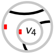
$R =$ 

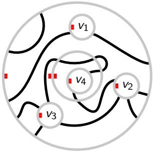
, $S =$ 

Linear maps

$$\mathsf{V}_T : P_4 \to P_8$$

$T =$ 

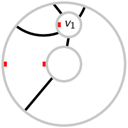
, $a =$ 

, $\mathsf{V}_T(a) =$ 

$\in P_8$

Action of annular vectors

Annular vectors

$R =$ 

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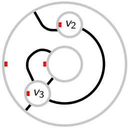
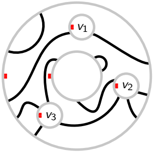
$S =$ 

Linear maps

$$V_T : P_4 \to P_8$$

$T =$ 

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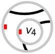
$a =$ 

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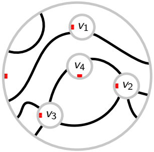
$V_T(a) =$ 

$\in P_8$

Discrete conformal nets

‘Conformal’ groups

$$
\begin{tikzcd}
\mathrm{Conf}(\mathbb{R}^{1,1}) \arrow[r, "\text{compactify}"] & \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \arrow[d, shift right, "\text{project}"'] \\
& \mathrm{Diff}_+(S^1) \arrow[u, dashed, shift right, "\text{reconstruct}"'] \arrow[d, shift right, "\text{discretise}"'] \\
& T_k \arrow[u, squiggly, shift right, "\text{cont. limit}"']
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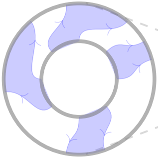
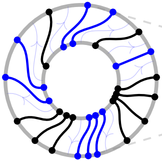
e.g.





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i) a Hilbert space $\mathcal{H}$

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iii) a continuous unitary representation U of $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$

Subject to:

Isotony:	$\mathcal{A}(I) \subseteq \mathcal{A}(J)$	if $I \subseteq J$
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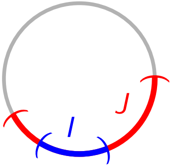

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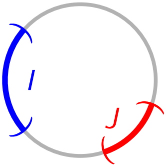

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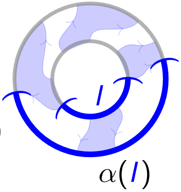

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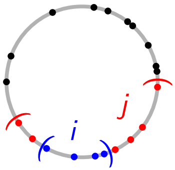

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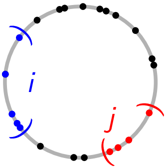

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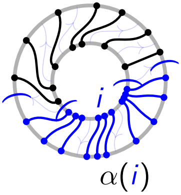

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A **semicontinuous model** (Jones '14) consists of:

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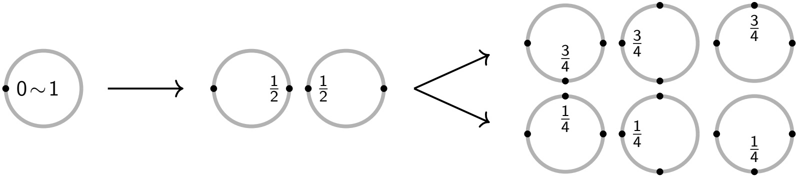
$\mathcal{D}$ is the set of k-adic subdivisions of S^1 for $k \in \mathbb{N}_{\geq 2}$



where S^1 is divided into intervals of the form $[\frac{m}{k^n}, \frac{m+1}{k^n}]$ for $m, n \in \mathbb{N}_0$.

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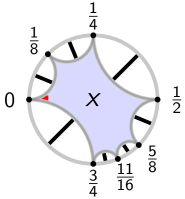
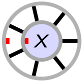
vectors in $\mathcal{H}_s$ are of the form:

$(s, x) \equiv$



$s = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{5}{8}, \frac{11}{16}, \frac{3}{4}) \in \mathcal{D}$

$x =$



$\in P_{|s|}$

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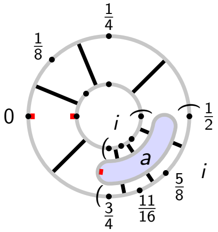
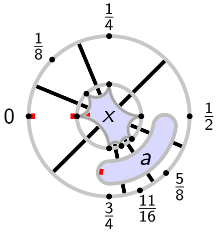
elements of $\mathcal{A}_s(i)$ are of the form:



action on $\mathcal{H}_s$



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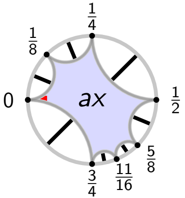

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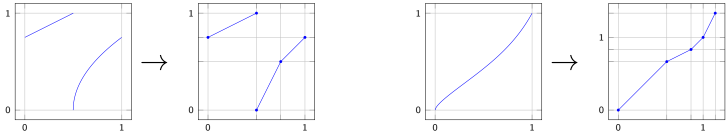
D is Thompson's group T_k



Theorem (Zhuang '07)

For each $f \in \mathrm{Diff}_+(S^1)$ there exists a $g \in T_k$ approx. f to arb. precision.

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Unitary representation induced by a map

$$\Phi : T_k \to (V_{m,n})_{m,n \in \mathbb{N}_0}$$

Continuity: work in progress!

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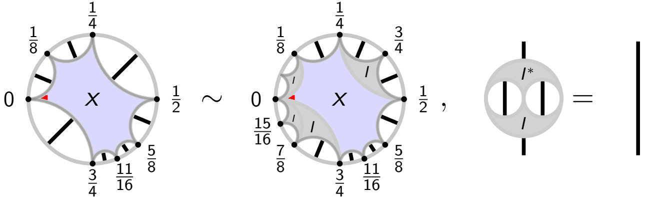
The full Hilbert space $\mathcal{H}$ is defined

$$\mathcal{H} := \bigcup_{s\in\mathcal{D}} \mathcal{H}_s / \sim,$$



where $R \in P_{k+1}$. The algebras $\mathcal{A}(i)$ are defined similarly.

Continuous representations

Definition

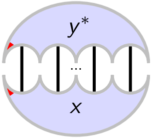
A representation π is *continuous* if each sequence $(f_n)_{n\in\mathbb{N}} \subset T_k$ satisfies

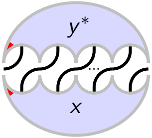
$$\lim_{n\to\infty} \|f_n - \mathrm{id}\| = 0, \qquad \lim_{n\to\infty} \langle x, \pi(f_n)(y)\rangle = \langle x, y\rangle, \qquad \forall\, x, y \in \mathcal{H}.$$

Denote by Rot_k the rotation subgroup of T_k, generated by:

$$\varrho_s : S^1 \to S^1, \qquad\qquad x \mapsto x + s \mod 1,$$

where s is a k-adic rational. Matrix elements can be expressed as:

$\langle x, y\rangle =$ 

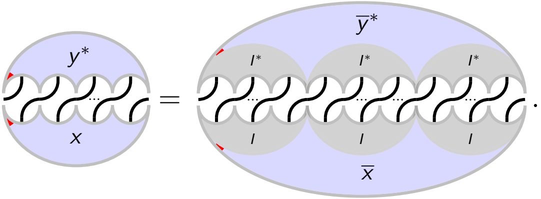
, $\qquad \langle x, U_I(\varrho_{\frac{1}{k^r}})(y)\rangle =$ 

where $x, y \in \mathcal{H}$.

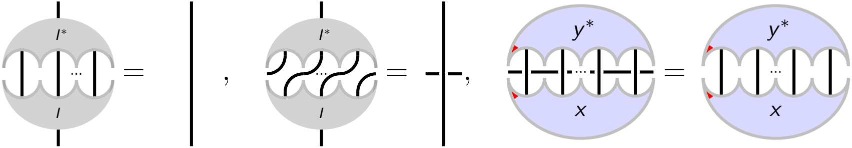
Continuity conditions

For each $x, y \in \mathcal{H}$, there exists a sufficiently large $r \in \mathbb{N}$, such that

$$\langle x, U_I(\varrho_{\frac{1}{k^r}})(y)\rangle =$$



If $I \in P_{k+1}$ is such that



for all $x, y \in \mathcal{H}$. Then

$$\lim_{r\to\infty} \langle x, U_I(\varrho_{\frac{1}{k^r}})(y)\rangle = \langle x, y\rangle,$$

and with other arguments $U_I : \mathrm{Rot}_k \to \mathrm{U}(\mathcal{H})$ is a continuous unitary rep.

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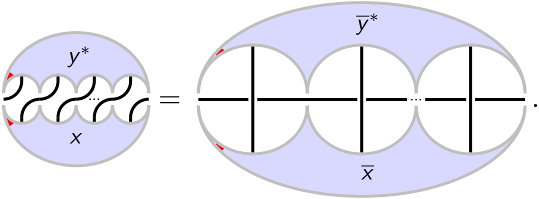

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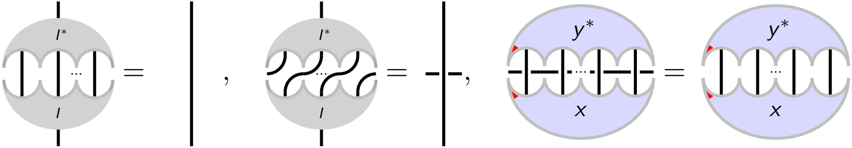

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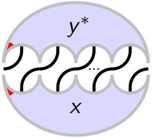
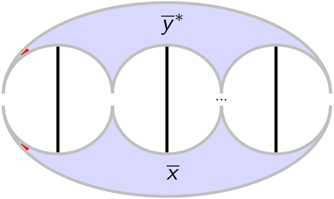
$$\left\langle x, U_I(\varrho_{\frac{1}{k^r}})(y)\right\rangle =$$ 

$=$ 

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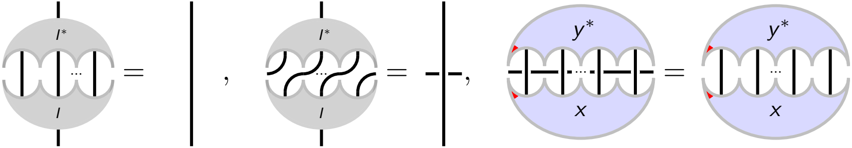

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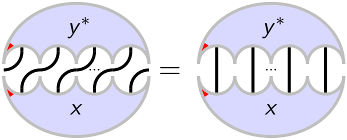

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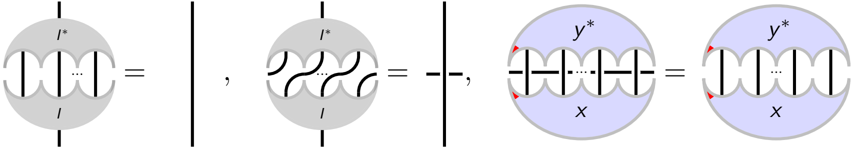

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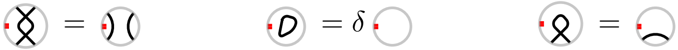
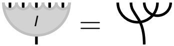
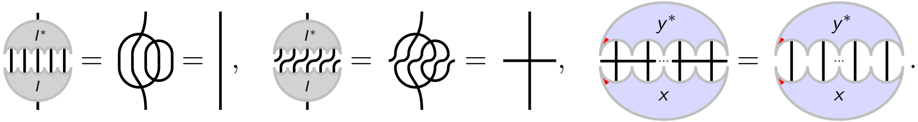
Brauer algebra solution

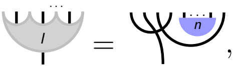
The Brauer planar algebra $(P_n)_{n\in 2\mathbb{N}_0}$ is generated by the action of planar tangles on the space $P_4 = \mathrm{span}(\{$  ,  ,  $\})$, subject to:



Specialising $k = 5$ and $\delta = 1$ we have the solution 



This solution can be generalised to $k = 2n + 5$ for all $n \in \mathbb{N}_0$



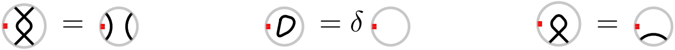
 $\in P_{2n}$.

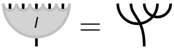
Theorem

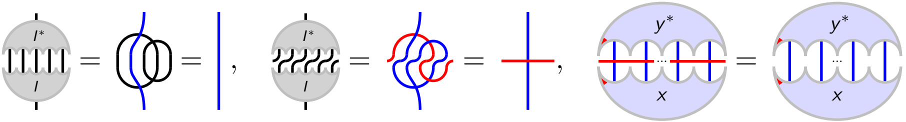
For $I \in P_{2n+6}$ above U_I is a continuous unitary representation of Rot_{2n+5}.

Brauer algebra solution

The Brauer planar algebra $(P_n)_{n \in 2\mathbb{N}_0}$ is generated by the action of planar tangles on the space $P_4 = \mathrm{span}(\{$  $\})$, subject to:



Specialising $k = 5$ and $\delta = 1$ we have the solution 



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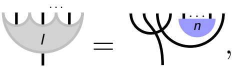

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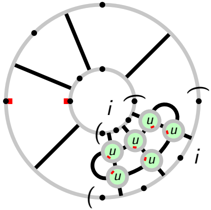
Integrable operators

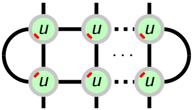
Spin chains on spacetime

R-**operators**: $u \in \mathbb{C}$

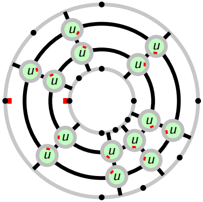
$R(u) =$  $= \sum_{a \in B_4} r_a(u)\, a$

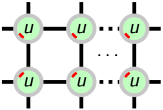
Local transfer operators

 $\in \mathcal{A}_s(i)$,

$L_n(u) :=$ 

Global transfer operators

 $\in \mathcal{A}_s$,

$G_n(u) :=$ 

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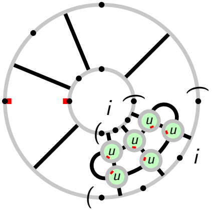

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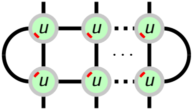
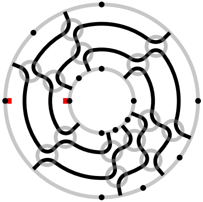
$L_n(u) :=$ 

Global transfer operators: e.g spacetime translations are generated by


$\in \mathcal{A}_s$

Integrability

A transfer operator $T(u) \in \mathcal{A}_s(i)$ is **integrable** if

$$[T(u), T(v)] = 0, \qquad \forall\, u, v \in \Omega \subseteq \mathbb{C}.$$

Expanding $T(u)$ in a basis of scalar functions

$$T(u) = \sum_{i=0}^{\infty} u^i Q_i, \quad \text{integrability implies} \quad [Q_i, Q_j] = 0, \quad \forall\, i, j \in \mathbb{N}_0,$$

where $H \equiv Q_1$ is the hamiltonian. $T(u)$ is **polynomially integrable** if

$$\boxed{\exists\, b \in \mathcal{A}_s(i) \;\text{ such that }\; T(u) \in \mathbb{C}(u)[b]}$$

Theorem (XP, Rasmussen '22)

If $T(u)$ is **integrable** and **diagonalisable** it is **polynomially integrable**.

Yang-Baxter integrability

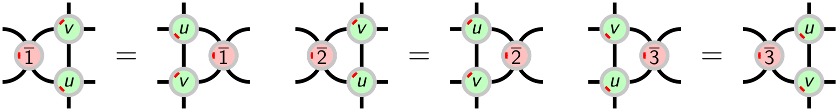
A model is **Yang-Baxter integrable** if the R-operators satisfy *local* relations that imply $[T(u), T(v)] = 0$.

For $T(u) \in \{L_n(u), G_n(u)\}$ a set of sufficient conditions is given by:

- Inversion identities

$(i = 1, 2, 3)$

- Yang-Baxter equations

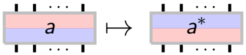


- Boundary Yang-Baxter equations

Polynomial integrability

There is an adjoint on each $\mathcal{A}_s(i)$ that acts as:

$$^* : \mathcal{A}_s(i) \to \mathcal{A}_s(i),$$



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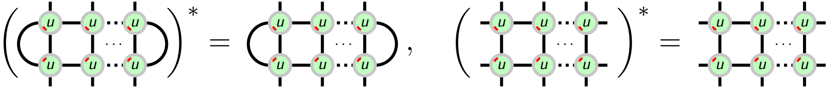

which is typically a property of **Yang-Baxter integrable** R-operators.

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For FC, BMW and Liu planar algebras if $T(u)$ is **Yang-Baxter integrable** then it is **polynomially integrable**.

The integrable structure of both local and global operators for such planar algebras is generated by a single spectral-independent operator!

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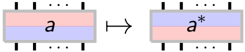

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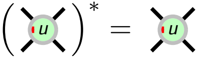

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Outlook

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Summary:

- Discrete conformal nets
- Planar algebras provide (almost) examples
- Continuity of representations remains a challenge
- A class of local and global operators are polynomially integrable

Future work:

- Continuity of representations for all of T_k
- More local and global operators
- Continuum limit taking discrete conformal nets to conformal nets
- Generalise to discrete algebraic quantum field theory

The end!