

# A planar-algebraic universe

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# Motivation

Some suggestions that spacetime is not a continuum:

- Aspects of the 'It from Qubit' program
- Successes of loop quantum gravity and causal sets
- Computational universe hypothesis

## Question

Can we discretise our theories such that they are recovered under a continuum limit?

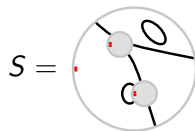
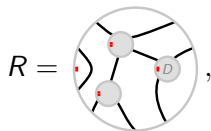
# Outline

- 1 Planar algebras
- 2 Discrete conformal nets
- 3 Semicontinuous models
- 4 Integrable operators
- 5 Outlook

# Planar algebras

# Planar algebras

## Planar tangles

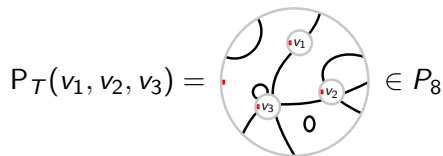
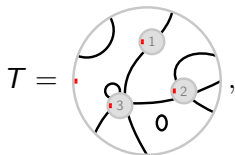


## Graded vector space

$$(P_n)_{n \in \mathbb{N}_0}$$

## Multilinear maps

$$P_T: P_2 \times P_4 \times P_6 \rightarrow P_8$$



# Example: Temperley-Lieb planar algebra

## Vector spaces

$$P_0 = \text{span}\left\{ \begin{array}{c} \text{red square} \\ \bigcirc \end{array} \right\}, \quad P_2 = \text{span}\left\{ \begin{array}{c} \text{red square} \\ \bigcirc \text{ with vertical line} \end{array} \right\}, \quad P_4 = \text{span}\left\{ \begin{array}{c} \text{red square} \\ \bigcirc \text{ with two arcs} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with two arcs (crossed)} \end{array} \right\},$$

$$P_6 = \text{span}\left\{ \begin{array}{c} \text{red square} \\ \bigcirc \text{ with three arcs} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with three arcs (crossed)} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with three arcs (crossed)} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with three arcs (crossed)} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with three arcs (crossed)} \end{array} \right\}, \quad \dots$$

## Planar tangle action

$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$

$$T = \begin{array}{c} \text{red square} \\ \bigcirc \text{ with three arcs labeled 1, 2, 3} \end{array}, \quad P_T\left( \begin{array}{c} \text{red square} \\ \bigcirc \text{ with two arcs} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with two arcs (crossed)} \end{array}, \begin{array}{c} \text{red square} \\ \bigcirc \text{ with two arcs (crossed)} \end{array} \right) = \begin{array}{c} \text{red square} \\ \bigcirc \text{ with complex tangle} \end{array}$$

# Example: Temperley-Lieb planar algebra

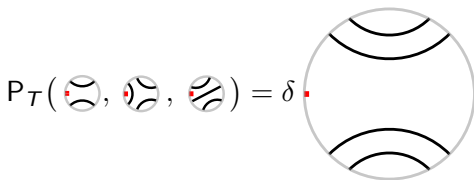
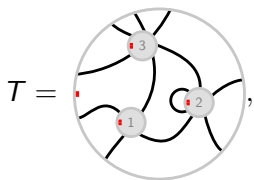
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$$P_6 = \text{span}\left\{ \begin{array}{c} \blacksquare \\ \bigcirc \left( \bigcirc \mid \bigcirc \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \bigcirc \left( \bigcirc \right) \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \bigcirc \left( \bigcirc \right) \right) \end{array}, \begin{array}{c} \blacksquare \\ \bigcirc \left( \bigcirc \left( \bigcirc \right) \right) \end{array}, \begin{array}{c} \blacksquare \\ \bigcirc \left( \bigcirc \left( \bigcirc \right) \right) \end{array} \right\}, \quad \dots$$

## Planar tangle action

$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$



# Example: Temperley-Lieb planar algebra

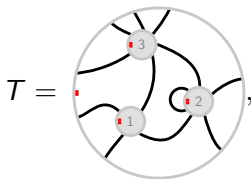
## Vector spaces

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$$P_6 = \text{span} \left\{ \begin{array}{c} \blacksquare \\ \bigcirc \left( \mid \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \diagup \diagdown \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \diagdown \diagup \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \left( \right) \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \left( \right) \right) \end{array} \right\}, \quad \dots$$

## Planar tangle action

$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$



$$P_T \left( \begin{array}{c} \blacksquare \\ \bigcirc \left( \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \left( \right) \right), \begin{array}{c} \blacksquare \\ \bigcirc \left( \diagup \diagdown \right) \right) = \delta \begin{array}{c} \blacksquare \\ \bigcirc \left( \left( \right) \right) \end{array} \end{array}$$



# Example: Tensor planar algebra

## Vector spaces

$$P_0 = \text{span} \left\{ \begin{array}{c} \blacksquare \\ \bigcirc \end{array} \right\}, \quad P_1 = \text{span} \left\{ \begin{array}{c} \blacksquare \\ \bigcirc \bullet_i \end{array} \mid i = 1, \dots, N \right\},$$

$$P_2 = \text{span} \left\{ \begin{array}{c} \bullet_i \\ \blacksquare \\ \bigcirc \\ \bullet_j \end{array} \mid i, j = 1, \dots, N \right\}, \quad P_3 = \text{span} \left\{ \begin{array}{c} \bullet_i \\ \blacksquare \\ \bigcirc \\ \bullet_j \\ \bullet_k \end{array} \mid i, j, k = 1, \dots, N \right\},$$

...

## Planar tangle action

$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$

$$T = \begin{array}{c} \blacksquare \\ \bigcirc \end{array}, \quad P_T \left( \begin{array}{c} \bullet_2 \\ \blacksquare \\ \bigcirc \\ \bullet_3 \\ \bullet_4 \end{array}, \begin{array}{c} \bullet_2 \\ \blacksquare \\ \bigcirc \\ \bullet_3 \\ \bullet_4 \end{array}, \begin{array}{c} \bullet_2 \\ \blacksquare \\ \bigcirc \\ \bullet_3 \\ \bullet_4 \end{array} \right) = \begin{array}{c} \blacksquare \\ \bigcirc \end{array}$$

# Example: Tensor planar algebra

## Vector spaces

$$P_0 = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \bigcirc \end{array} \right\}, \quad P_1 = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \bigcirc \bullet_i \end{array} \mid i = 1, \dots, N \right\},$$

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...

## Planar tangle action

$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$

$$T = \begin{array}{c} \bigcirc \\ \text{tangle diagram} \end{array}, \quad P_T \left( \begin{array}{c} \text{tangle 1} \\ \text{tangle 2} \\ \text{tangle 3} \end{array} \right) = \begin{array}{c} \bigcirc \\ \text{resulting tangle} \end{array}$$

The diagram shows a large circle containing a tangle with three red squares labeled 1, 2, and 3. To the right, the action of the planar tangle  $T$  is shown as a composition of three smaller tangles (each with a red square) and a larger resulting tangle with eight red squares labeled 1 through 8.

# Example: Tensor planar algebra

## Vector spaces

$$P_0 = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \end{array} \right\}, \quad P_1 = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \bullet_i \end{array} \mid i = 1, \dots, N \right\},$$

$$P_2 = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \bullet_i \\ \bullet_j \end{array} \mid i, j = 1, \dots, N \right\}, \quad P_3 = \text{span} \left\{ \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \bullet_i \\ \bullet_j \\ \bullet_k \end{array} \mid i, j, k = 1, \dots, N \right\},$$

...

## Planar tangle action

$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$

$$T = \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \text{tangle} \end{array}, \quad P_T \left( \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \bullet_2 \bullet_3 \\ \bullet_3 \bullet_4 \end{array}, \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \bullet_1 \bullet_2 \bullet_3 \\ \bullet_1 \bullet_4 \end{array}, \begin{array}{c} \text{red square} \\ \circlearrowleft \\ \bullet_1 \bullet_2 \bullet_4 \\ \bullet_4 \bullet_3 \end{array} \right) = \begin{array}{c} \bullet_1 \bullet_2 \\ \circlearrowleft \\ \bullet_4 \bullet_3 \\ \bullet_2 \bullet_1 \end{array}$$

# Example: Tensor planar algebra

## Vector spaces

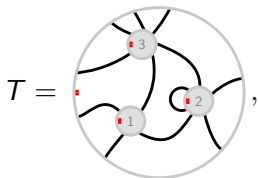
$$P_0 = \text{span} \left\{ \begin{array}{c} \blacksquare \\ \bigcirc \end{array} \right\}, \quad P_1 = \text{span} \left\{ \begin{array}{c} \blacksquare \bullet_i \\ \bigcirc \end{array} \mid i = 1, \dots, N \right\},$$

$$P_2 = \text{span} \left\{ \begin{array}{c} \bullet_i \\ \blacksquare \\ \bullet_j \\ \bigcirc \end{array} \mid i, j = 1, \dots, N \right\}, \quad P_3 = \text{span} \left\{ \begin{array}{c} \bullet_i \\ \blacksquare \\ \bullet_j \\ \bullet_k \\ \bigcirc \end{array} \mid i, j, k = 1, \dots, N \right\},$$

...

## Planar tangle action

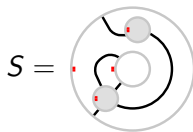
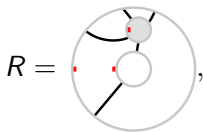
$$P_T: P_4 \times P_6 \times P_6 \rightarrow P_8$$



$$P_T \left( \begin{array}{c} 2 \\ \blacksquare \\ 3 \end{array} \begin{array}{c} 3 \\ \bullet \\ 4 \end{array}, \begin{array}{c} 1 \\ \blacksquare \\ 4 \end{array} \begin{array}{c} 3 \\ \bullet \\ 1 \end{array}, \begin{array}{c} 1 \\ \blacksquare \\ 3 \end{array} \begin{array}{c} 4 \\ \bullet \\ 2 \end{array} \right) = \begin{array}{c} 1 \quad 2 \\ \blacksquare \\ 4 \quad 3 \\ \bullet \\ 2 \quad 3 \\ \bullet \\ 3 \quad 1 \end{array}$$

# Annular tangles

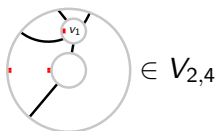
## Annular tangles



## Graded annular vector space

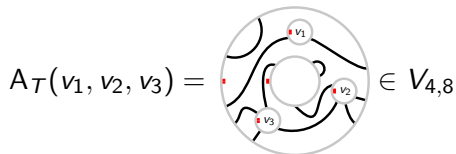
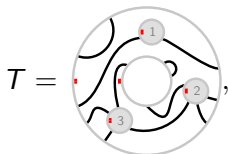
$$(V_{m,n})_{m,n \in \mathbb{N}_0}$$

e.g.



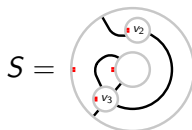
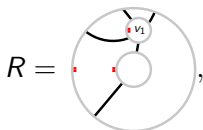
## Multilinear maps

$$A_T: P_2 \times P_4 \times P_4 \rightarrow V_{4,8}$$



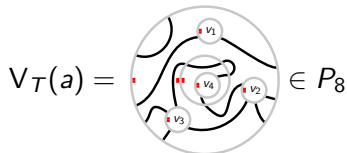
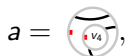
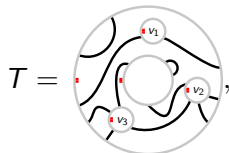
# Action of annular vectors

## Annular vectors



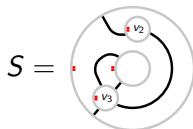
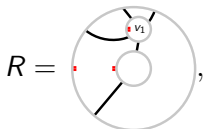
## Linear maps

$$V_T: P_4 \rightarrow P_8$$



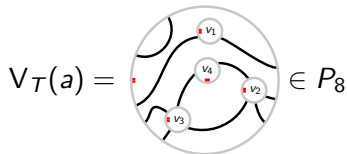
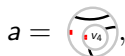
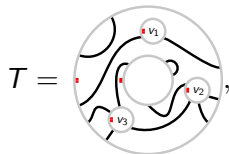
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## Annular vectors



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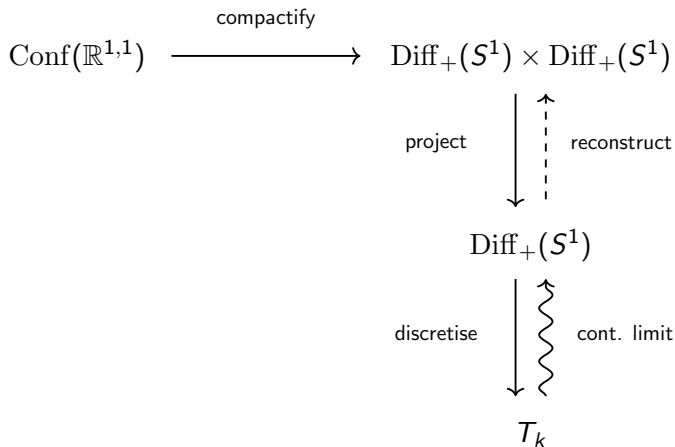
$$V_T: P_4 \rightarrow P_8$$



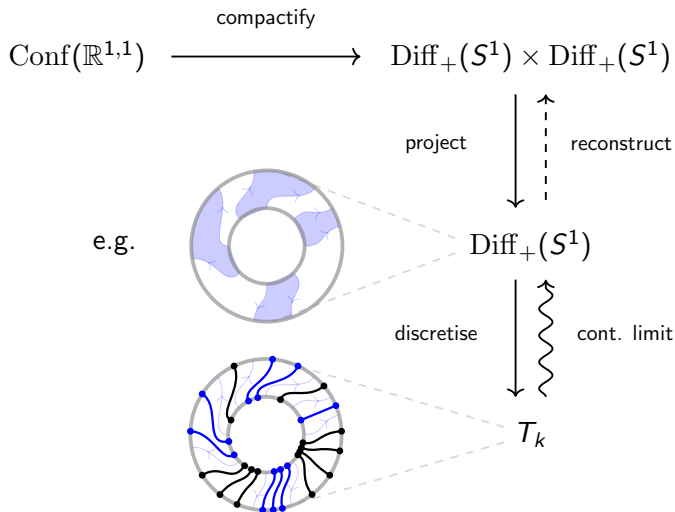
## Discrete conformal nets



## 'Conformal' groups



# 'Conformal' groups



# Conformal nets

A **conformal net** (Haag-Kastler '64) consists of:

- i) a Hilbert space  $\mathcal{H}$
- ii) a  $C^*$ -algebra  $\mathcal{A}(I)$  on  $\mathcal{H}$  for each open interval  $I \subset S^1$
- iii) a continuous unitary representation  $U$  of  $\text{Diff}_+(S^1)$  on  $\mathcal{H}$

Subject to:

$$\text{Isotony: } \mathcal{A}(I) \subseteq \mathcal{A}(J) \quad \text{if } I \subseteq J$$

$$\text{Locality: } [\mathcal{A}(I), \mathcal{A}(J)] = 0 \quad \text{if } I \cap J = \emptyset$$

$$\text{Covariance: } U(\alpha)\mathcal{A}(I)U(\alpha)^* = \mathcal{A}(\alpha(I)) \quad \alpha \in \text{Diff}_+(S^1)$$

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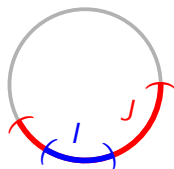
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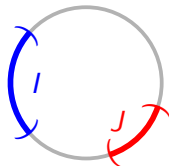
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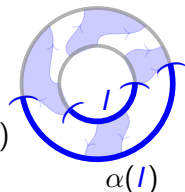
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# Discrete conformal nets

A **discrete conformal net** consists of:

- i) a directed set  $\mathcal{D}$  of finite subsets of  $S^1$
- ii) a Hilbert space  $\mathcal{H}_s$  for each  $s \in \mathcal{D}$
- iii) a  $C^*$ -algebra  $\mathcal{A}_s(i)$  on  $\mathcal{H}_s$  for each connected  $i \subset s$  and  $s \in \mathcal{D}$
- iv) a discrete realisation of  $\text{Diff}_+(S^1)$  denoted  $\mathbb{D}$
- v) a continuous unitary representation  $U$  of  $\mathbb{D}$  on  $\mathcal{H}$  (see below)

The full Hilbert space  $\mathcal{H}$  and  $C^*$ -algebras  $\mathcal{A}(i)$  are constructed from 'direct limits' of  $\mathcal{H}_s$  and  $\mathcal{A}_s(i)$  respectively. Subject to:

$$\text{Isotony: } \mathcal{A}(i) \subseteq \mathcal{A}(j) \quad \text{if } i \subseteq j$$

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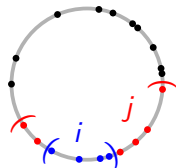
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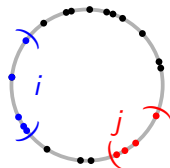
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The full Hilbert space  $\mathcal{H}$  and  $C^*$ -algebras  $\mathcal{A}(i)$  are constructed from 'direct limits' of  $\mathcal{H}_s$  and  $\mathcal{A}_s(i)$  respectively. Subject to:

$$\text{Isotony: } \mathcal{A}(i) \subseteq \mathcal{A}(j) \quad \text{if } i \subseteq j$$

$$\text{Locality: } [\mathcal{A}(i), \mathcal{A}(j)] = 0 \quad \text{if } i \cap j = \emptyset$$

$$\text{Covariance: } U(\alpha)\mathcal{A}(i)U(\alpha)^* = \mathcal{A}(\alpha(i)) \quad \alpha \in \mathbb{D}$$



# Discrete conformal nets

A **discrete conformal net** consists of:

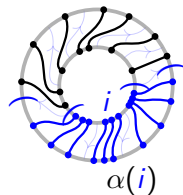
- i) a directed set  $\mathcal{D}$  of finite subsets of  $S^1$
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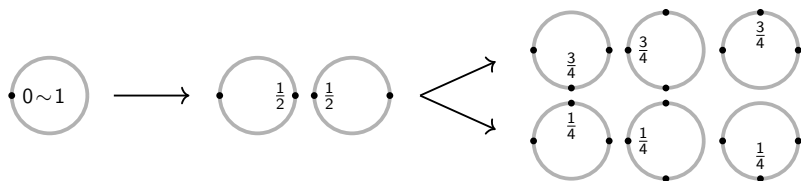
## Semicontinuous models

# Semicontinuous models

A **semicontinuous model** (Jones '14) consists of:

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- iv) a discrete realisation of  $\text{Diff}_+(S^1)$  denoted  $\mathbb{D}$
- v) a continuous unitary representation  $U$  of  $\mathbb{D}$  on  $\mathcal{H}$

$\mathcal{D}$  is the set of  $k$ -adic subdivisions of  $S^1$  for  $k \in \mathbb{N}_{\geq 2}$



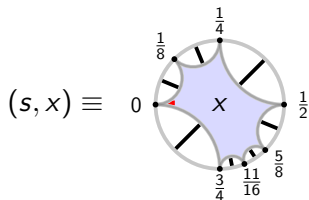
where  $S^1$  is divided into intervals of the form  $[\frac{m}{k^n}, \frac{m+1}{k^n}]$  for  $m, n \in \mathbb{N}_0$ .

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vectors in  $\mathcal{H}_s$  are of the form:



$$s = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{5}{8}, \frac{11}{16}, \frac{3}{4}) \in \mathcal{D}$$

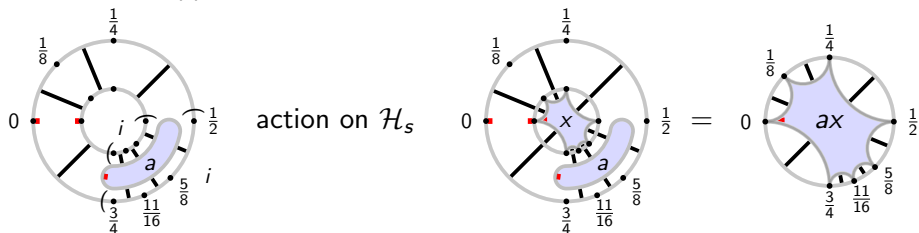
$$x = \text{[Diagram of a circle with a shaded region X and a red dot]} \in P_{|s|}$$

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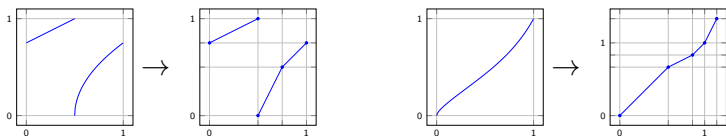


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$\mathbb{D}$  is Thompson's group  $T_k$



**Theorem (Zhuang '07)**

For each  $f \in \text{Diff}_+(S^1)$  there exists a  $g \in T_k$  approx.  $f$  to arb. precision.

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- v) a **continuous** unitary representation  $U$  of  $D$  on  $\mathcal{H}$

Unitary representation induced by a map

$$\Phi : T_k \rightarrow (V_{m,n})_{m,n \in \mathbb{N}_0}$$

**Continuity:** work in progress!



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- iv) a discrete realisation of  $\text{Diff}_+(S^1)$  denoted  $D$ ;
- v) a **continuous** unitary representation  $U$  of  $D$  on  $\mathcal{H}$

The full Hilbert space  $\mathcal{H}$  is defined

$$\mathcal{H} := \bigcup_{s \in \mathcal{D}} \mathcal{H}_s / \sim,$$

where  $R \in P_{k+1}$ . The algebras  $\mathcal{A}(i)$  are defined similarly.

# Continuous representations

## Definition

A representation  $\pi$  is *continuous* if each sequence  $(f_n)_{n \in \mathbb{N}} \subset T_k$  satisfies

$$\lim_{n \rightarrow \infty} \|f_n - \text{id}\| = 0, \quad \lim_{n \rightarrow \infty} \langle x, \pi(f_n)(y) \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathcal{H}.$$

Denote by  $\text{Rot}_k$  the rotation subgroup of  $T_k$ , generated by:

$$\varrho_s : S^1 \rightarrow S^1, \quad x \mapsto x + s \pmod{1},$$

where  $s$  is a  $k$ -adic rational. Matrix elements can be expressed as:

$$\langle x, y \rangle = \text{Diagram 1}, \quad \langle x, U_l(\varrho_{\frac{1}{k^l}})(y) \rangle = \text{Diagram 2}$$

where  $x, y \in \mathcal{H}$ .

# Continuity conditions

For each  $x, y \in \mathcal{H}$ , there exists a sufficiently large  $r \in \mathbb{N}$ , such that

$$\langle x, U_I(\varrho_{\frac{1}{k^r}})(y) \rangle = \text{Diagram 1} = \text{Diagram 2}$$

The diagram shows two representations of the inner product. The first is a circle with a wavy line representing the interaction  $I$ . The top half is shaded light blue and labeled  $y^*$ , and the bottom half is shaded light blue and labeled  $x$ . The second is a larger oval with three smaller circles inside, each containing a wavy line. The top half is shaded light blue and labeled  $\bar{y}^*$ , and the bottom half is shaded light blue and labeled  $\bar{x}$ . The three inner circles are shaded light gray and each labeled  $I^*$ . Red arrows point from the wavy lines in the inner circles towards the top and bottom boundaries.

If  $I \in P_{k+1}$  is such that

$$\text{Diagram 3} = \text{Diagram 4}, \quad \text{Diagram 5} = \text{Diagram 6}, \quad \text{Diagram 7} = \text{Diagram 8}$$

The diagram shows three pairs of equalities. The first pair shows a circle with a wavy line and a vertical line passing through it, equal to a vertical line. The second pair shows a circle with a wavy line and a vertical line passing through it, equal to a vertical line with a wavy line. The third pair shows a circle with a wavy line and a vertical line passing through it, equal to a circle with a wavy line. The top half of the circles is shaded light blue and labeled  $y^*$ , and the bottom half is shaded light blue and labeled  $x$ . Red arrows point from the wavy lines towards the top and bottom boundaries.

for all  $x, y \in \mathcal{H}$ . Then

$$\lim_{r \rightarrow \infty} \langle x, U_I(\varrho_{\frac{1}{k^r}})(y) \rangle = \langle x, y \rangle,$$

and with other arguments  $U_I : \text{Rot}_k \rightarrow U(\mathcal{H})$  is a continuous unitary rep.

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Diagram 1: A circular region with a wavy boundary. The interior is shaded light blue. The top boundary is labeled  $y^*$  and the bottom boundary is labeled  $x$ . Red arrows point inward from the wavy boundary.

Diagram 2: A larger, horizontally elongated region with a smooth boundary. The interior is shaded light blue. The top boundary is labeled  $\bar{y}^*$  and the bottom boundary is labeled  $\bar{x}$ . Inside, there are three vertical lines representing a simplified wavy boundary. Red arrows point inward from the outer boundary.

If  $I \in P_{k+1}$  is such that

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Diagram 3: A circular region with a smooth boundary. The interior is shaded gray. The top boundary is labeled  $I^*$  and the bottom boundary is labeled  $I$ . Inside, there are three vertical lines.

Diagram 4: A vertical line.

Diagram 5: A circular region with a wavy boundary. The interior is shaded gray. The top boundary is labeled  $I^*$  and the bottom boundary is labeled  $I$ . Inside, there are three wavy lines.

Diagram 6: A vertical line.

Diagram 7: A circular region with a wavy boundary. The interior is shaded light blue. The top boundary is labeled  $y^*$  and the bottom boundary is labeled  $x$ . Inside, there are three vertical lines. Red arrows point inward from the wavy boundary.

Diagram 8: A circular region with a smooth boundary. The interior is shaded light blue. The top boundary is labeled  $y^*$  and the bottom boundary is labeled  $x$ . Inside, there are three vertical lines. Red arrows point inward from the smooth boundary.

for all  $x, y \in \mathcal{H}$ . Then

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# Brauer algebra solution

The Brauer planar algebra  $(P_n)_{n \in 2\mathbb{N}_0}$  is generated by the action of planar tangles on the space  $P_4 = \text{span}(\{ \text{⊙}, \text{⊖}, \text{⊗} \})$ , subject to:

$$\text{⊗} = \text{⊙} \quad \text{⊖} = \delta \text{⊙} \quad \text{⊙} = \text{⊖}$$

Specialising  $k = 5$  and  $\delta = 1$  we have the solution  $\text{⊖} = \Psi$

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This solution can be generalised to  $k = 2n + 5$  for all  $n \in \mathbb{N}_0$

$$\text{⊖} = \text{⊖}, \quad \text{⊙} \in P_{2n}.$$

## Theorem

For  $I \in P_{2n+6}$  above  $U_I$  is a continuous unitary representation of  $\text{Rot}_{2n+5}$ .

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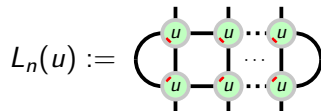
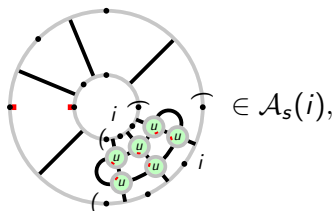
# Integrable operators

## Spin chains on spacetime

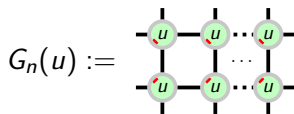
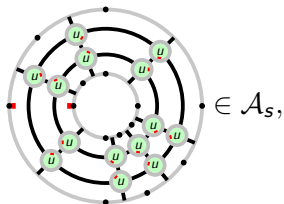
 $R$ -operators:  $u \in \mathbb{C}$ 

$$R(u) = \text{diag}(u) = \sum_{a \in B_4} r_a(u) a$$

Local transfer operators



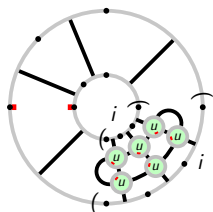
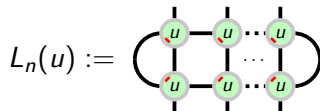
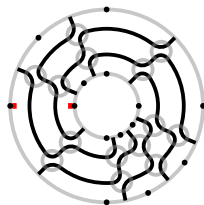
Global transfer operators



## Spin chains on spacetime

**R-operators:**  $u \in \mathbb{C}$ 

$$R(u) = \text{diag}(u) = \sum_{a \in B_4} r_a(u) a$$

**Local transfer operators** $\in \mathcal{A}_S(i),$  $L_n(u) :=$ **Global transfer operators:** e.g spacetime translations are generated by $\in \mathcal{A}_S$

# Integrability

A transfer operator  $T(u) \in \mathcal{A}_s(i)$  is **integrable** if

$$[T(u), T(v)] = 0, \quad \forall u, v \in \Omega \subseteq \mathbb{C}.$$

Expanding  $T(u)$  in a basis of scalar functions

$$T(u) = \sum_{i=0}^{\infty} u^i Q_i, \quad \text{integrability implies } [Q_i, Q_j] = 0, \quad \forall i, j \in \mathbb{N}_0,$$

where  $H \equiv Q_1$  is the hamiltonian.  $T(u)$  is **polynomially integrable** if

$$\exists b \in \mathcal{A}_s(i) \text{ such that } T(u) \in \mathbb{C}(u)[b]$$

Theorem (XP, Rasmussen '22)

If  $T(u)$  is **integrable** and **diagonalisable** it is **polynomially integrable**.

# Yang-Baxter integrability

A model is **Yang-Baxter integrable** if the  $R$ -operators satisfy *local* relations that imply  $[T(u), T(v)] = 0$ .

For  $T(u) \in \{L_n(u), G_n(u)\}$  a set of sufficient conditions is given by:

- Inversion identities

$$\begin{array}{c} \textcircled{i} \\ \textcircled{\bar{i}} \end{array} = \text{---} = \begin{array}{c} \textcircled{\bar{i}} \\ \textcircled{i} \end{array} \quad (i = 1, 2, 3)$$

- Yang-Baxter equations

$$\begin{array}{c} \textcircled{\bar{1}} \\ \textcircled{v} \\ \textcircled{u} \end{array} = \begin{array}{c} \textcircled{u} \\ \textcircled{\bar{1}} \\ \textcircled{v} \end{array} \quad \begin{array}{c} \textcircled{\bar{2}} \\ \textcircled{v} \\ \textcircled{u} \end{array} = \begin{array}{c} \textcircled{u} \\ \textcircled{\bar{2}} \\ \textcircled{v} \end{array} \quad \begin{array}{c} \textcircled{\bar{3}} \\ \textcircled{v} \\ \textcircled{u} \end{array} = \begin{array}{c} \textcircled{u} \\ \textcircled{\bar{3}} \\ \textcircled{v} \end{array}$$

- Boundary Yang-Baxter equations

$$\begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} = \begin{array}{c} \textcircled{3} \\ \textcircled{4} \end{array} \quad \begin{array}{c} \textcircled{\bar{1}} \\ \textcircled{2} \end{array} = \begin{array}{c} \textcircled{3} \\ \textcircled{\bar{4}} \end{array}$$

# Polynomial integrability

There is an adjoint on each  $\mathcal{A}_S(i)$  that acts as:

$$* : \mathcal{A}_S(i) \rightarrow \mathcal{A}_S(i),$$



For  $T(u) \in \{L_n(u), G_n(u)\}$  establishing **diagonalisability** amounts to:

which is typically a property of **Yang-Baxter integrable**  $R$ -operators.

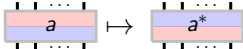
## Theorem (XP, Rasmussen '23)

For FC, BMW and Liu planar algebras if  $T(u)$  is **Yang-Baxter integrable** then it is **polynomially integrable**.

The integrable structure of both local and global operators for such planar algebras is generated by a single spectral-independent operator!

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$$\left( \begin{array}{c} \diagup \quad \diagdown \\ \textcircled{u} \\ \diagdown \quad \diagup \end{array} \right)^* = \begin{array}{c} \diagdown \quad \diagup \\ \textcircled{u} \\ \diagup \quad \diagdown \end{array}$$

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The integrable structure of both local and global operators for such planar algebras is generated by a single spectral-independent operator!

# Outlook



# Outlook

## Summary:

- Discrete conformal nets
- Planar algebras provide (almost) examples
- Continuity of representations remains a challenge
- A class of local and global operators are polynomially integrable

## Future work:

- Continuity of representations for all of  $T_k$
- More local and global operators
- Continuum limit taking discrete conformal nets to conformal nets
- Generalise to discrete algebraic quantum field theory

The end!