Loop models and triangulations

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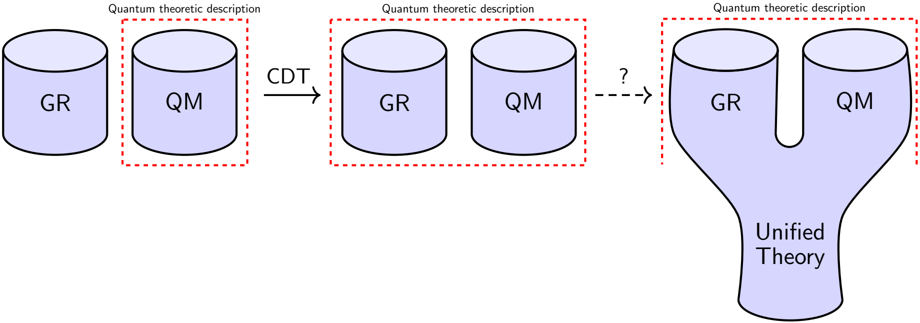
Confirmation seminar, April 2020

Outline

Introduction

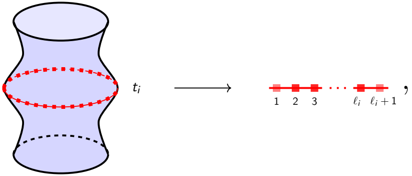
Motivation



GR - General Relativity
QM - Quantum Mechanics
CDT - Causal Dynamical Triangulations

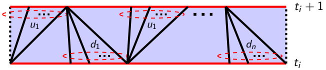
CDT I

We will be considering a $1+1$-dimensional CDT model with the topology $S^1 \times \mathbb{N}^+$. An example CDT universe is given by

Triangulation example


, where $\ell_i + 1 \equiv 1$.

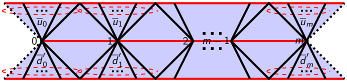
Infinitesimal time steps $t_i \to t_i + 1$ are generated by the *transfer matrix*

$$\mathbb{T} = \sum_{i, u_i, d_i, n}$$


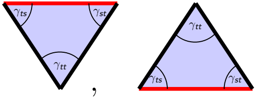
which is uniquely defined by the sequences $\mathbf{u}$ and $\mathbf{d}$.

CDT II

The multiplication of two transfer matrices generate all possible two-instant triangulations

$$\mathbb{T}^2 = \sum_{i,\overline{u}_i,\overline{d}_i,m}$$ 

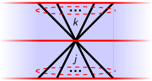
We consider the CDT universe to be generated by $\mathbb{T}^N$, where N denotes the number of time instants in the universe. Let us now consider the structure of the constituent triangles:



– space-like – time-like

These objects are flat in the sense that $\gamma_{ts} + \gamma_{st} + \gamma_{tt} = \pi$.

Let us consider a node with j incoming and k outgoing edges



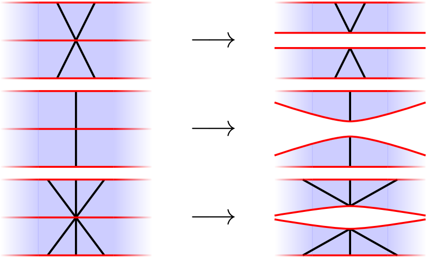
$$, \qquad 2\pi + (k + j - 4)\gamma_{tt} = \Lambda.$$

Despite appearance, the internal structure of the triangles is maintained. This gives rise to non-Euclidean geometries

Euclidean : $j = k = 2, \quad \Lambda = 2\pi$

Hyperbolic : $j = k = 1, \quad \Lambda < 2\pi$

Elliptic : $j = k = 3, \quad \Lambda > 2\pi$

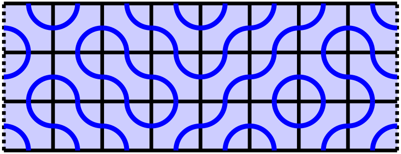


Loop models I

Of seemingly independent interest are lattice loop models (LLM), providing descriptions of systems possessing non-local degrees of freedom:

- Percolation
- Spin clusters
- Polymer chains

Let us consider a particular lattice configuration on the cylinder

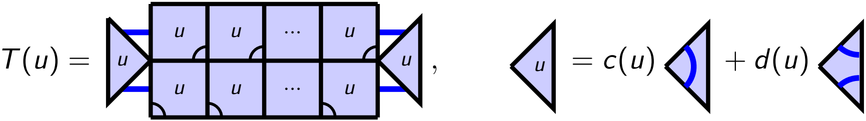


Naturally occurring within the lattice are contractible loops. These objects are assigned a non-local parameter β.

Loop models II

We can describe of these models with a transfer matrix

For a general class of boundary conditions we have a double-row object



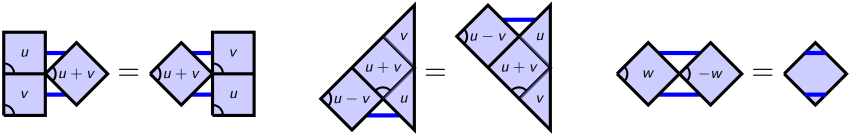
Here we interpret the coefficients $a(u)$, $b(u)$, $c(u)$ and $d(u)$ as local Boltzmann weights.

Loop models III

We seek to fix the local and non-local weights such that the model is endowed with the property of integrability. The integrability of a lattice model is encoded in the transfer matrix

$$[T(u), T(v)] = 0 \quad \Longrightarrow \quad \text{(infinite) set of conserved quantities}$$

Sufficient conditions for integrability:



Loop models on triangulations

- Theories of quantum gravity incorporate gravity-matter couplings
- Raw triangulations encode the geometry of the gravitational field
- Loop models describe interaction of matter fields
- Coupling matter to triangulations we assign a loop degree of freedom to each simplex
- We seek to assign this freedom in a way that maintains integrability

Project structure

Project structure

LLM:

- ◇ Braid monoid models
- ◇ Fusion procedure
- ◇ Boundary conditions
- ◇ Link invariants

Integrability

- - - - →

- - - - →

- - - - →

Framework

CDT:

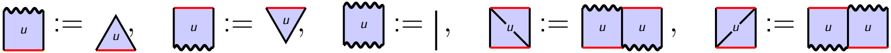
- ◇ t-Braid monoid models
- ◇ t-Fusion procedure
- ◇ t-Boundary conditions

Progress and methods

Integrability I

Here we present the *dense model*

Typical notions of integrability are established on a regular square lattice:

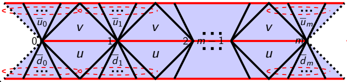


This is accompanied by vertical multiplication rules, encoding the underlying "triangular nature" of these objects. Finally we introduce

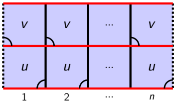
where horizontal multiplication rules eliminate under/overcounting

Integrability II

$\mathbb{T}(u)\mathbb{T}(v) = \sum_{\overline{u}_i,\overline{d}_i,m}$

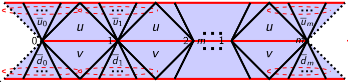


$\longleftrightarrow \sum_{n=1}^{\infty}$

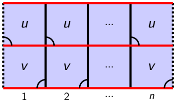


$\|$

$\mathbb{T}(v)\mathbb{T}(u) = \sum_{\overline{u}_i,\overline{d}_i,m}$



$\longleftrightarrow \sum_{n=1}^{\infty}$



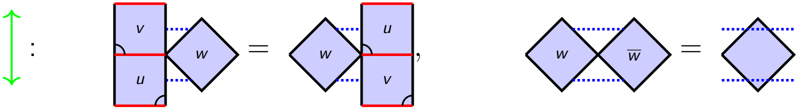
Establishing the red and green connections, we conclude $[\mathbb{T}(u), \mathbb{T}(v)] = 0$

$\longleftrightarrow$ Horizontal/vertical multiplication rules

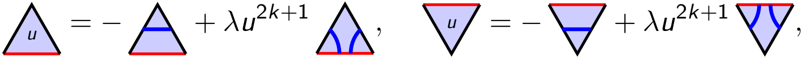
$\updownarrow$ Integrability

Integrability III

Similar to regular lattice models, sufficient conditions for integrability are



Any model satisfying these conditions is integrable. For the particular case of the dense model, we have found the solution $w = uv$, $\overline{w} = -uv$

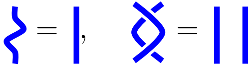
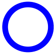
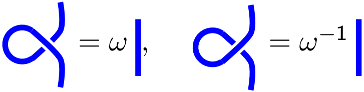


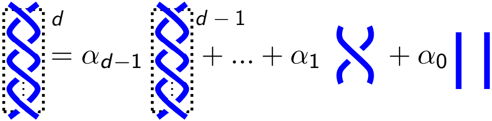
$\beta = 0$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ are free parameters of the model.

Braid monoid algebras I

Our goal is to develop a generalisation of the so-called braid monoid algebra as a quotient of the braid group algebra. Distilling the properties of crossing loop segments we arrive at the four rules

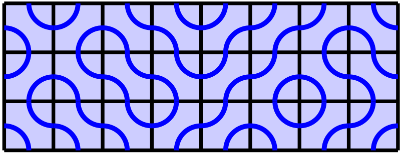
1. Invariance under regular isotopy: 
2. Contractible loops are removed:  $= \beta$
3. Twists are removed: 

4. Twisting limit of d: 

These properties are encoded as quotients of the braid group algebra.

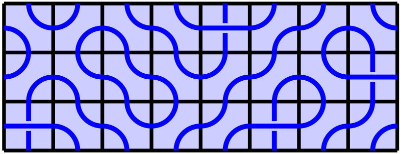
Braid monoid algebras II

For $d = 2$ we have Temperley-Lieb-like algebras $\mathrm{TL}_n(\omega)$



, $\beta = \{\pm 1, -\omega - \omega^{-1}\}$

For $d = 3$ we have Birman–Murakami–Wenzl-like algebras $\mathrm{BMW}_n(\omega, r)$

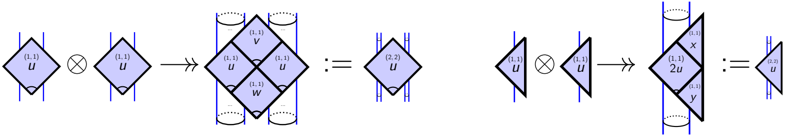


, $$\beta = \{\pm 1, -\omega - \omega^{-1}, \pm i, \frac{(\omega/r - 1)(\omega r \pm 1)}{\omega^2 \mp 1}\}$$

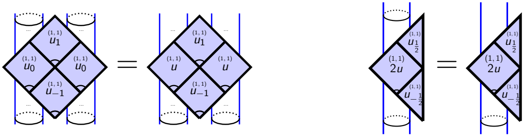
These solutions persist up to arbitrary d.

Fusion

The fusion procedure allows the construction of higher spin systems



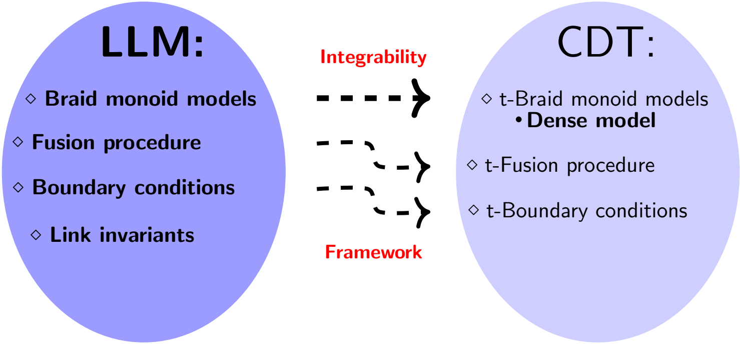
Exploiting the integrability of the underlying model we can endow the fused model with this property. It suffices to satisfy the *drop-down* condition



where $u_m = u + m\lambda$

Selecting $v = u_1$, $w = u_{-1}$, $x = u_{1/2}$ and $y = u_{-1/2}$, the drop-down property holds for both the bulk and boundary.

Road map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◇ Fusion procedure
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◇ Link invariants
Integrability
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