Loop models on causal triangulations

J. Stat. Mech. (2021) 113102

Xavier Poncini, PhD candidate UQ

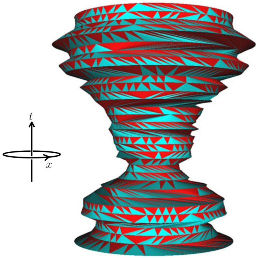
in collaboration with: Bergfinnur Durhuus, Jørgen Rasmussen and Meltem Ünel

ANZAMP 2022

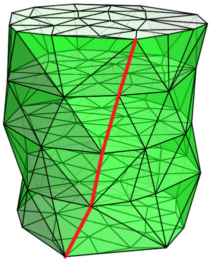
Causal dynamical triangulations

Causal dynamical triangulations (CDT) is an approach to quantum gravity that offers a tractable way to compute the path integral

$$\langle F \rangle = \int_{x_i \to x_f} \mathcal{D}\mathbf{g}\mathcal{D}\varphi F[\mathbf{g}, \varphi] e^{iS[\mathbf{g},\varphi]} \to \sum_{\mathcal{T},\varphi} F[\mathcal{T}, \varphi] e^{iS[\mathcal{T},\varphi]}$$



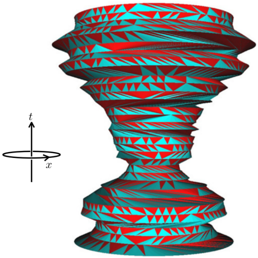
$1+1$-dimensional
[Israiel and Linder 12']



$2+1$-dimensional
[Budd 12']

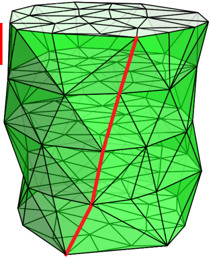
Causal dynamical triangulations (CDT) is an approach to quantum gravity that offers a tractable way to compute the path integral

$$\langle F \rangle = \int_{x_i \to x_f} \mathcal{D}\mathbf{g}\mathcal{D}\varphi F[\mathbf{g}, \varphi] e^{iS[\mathbf{g},\varphi]} \to \sum_{\mathcal{T},\varphi} F[\mathcal{T}, \varphi] e^{iS[\mathcal{T},\varphi]}$$



1 + 1-dimensional
[Israiel and Linder 12']

Work in progress!

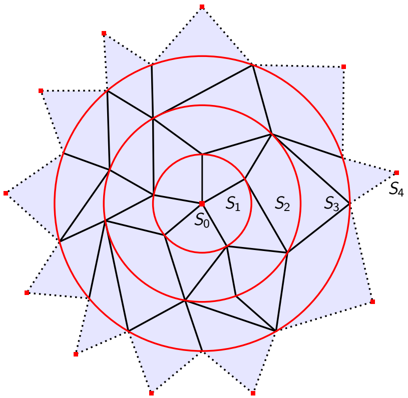


2 + 1-dimensional
[Budd 12']

Outline

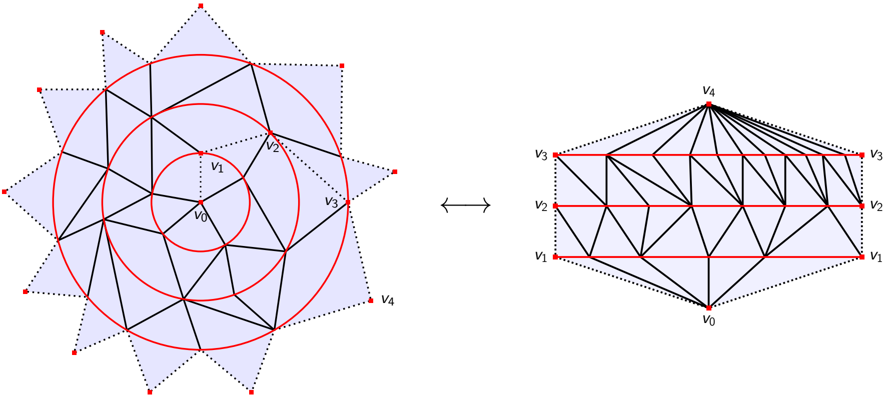
Pure CDT model

A causal triangulation of a sphere is defined by a sequence of circular graphs $S_0, S_1, \ldots, S_m, S_{m+1}$, where $m \in \mathbb{N}$ is the height



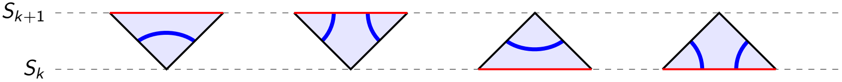
such that the annulus between two cycles is triangulated. Space-like edges are coloured red, while time-like edges are coloured black.

Each triangulation of the sphere admits a unique map to the plane defined by the sequence $v_0, v_1, \ldots, v_m, v_{m+1}$:

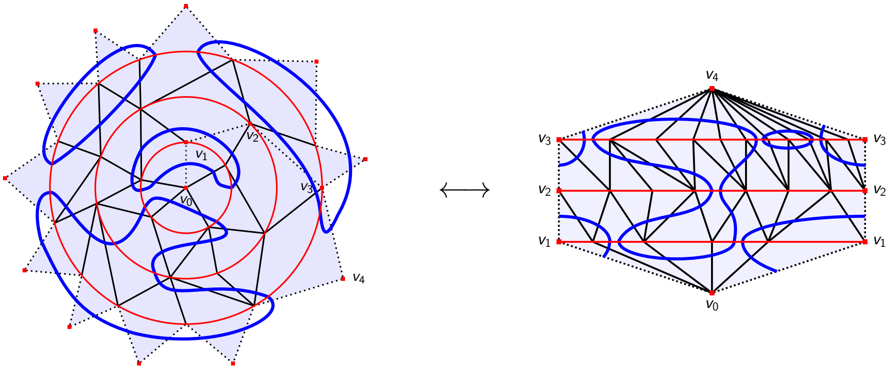


Dense loop model

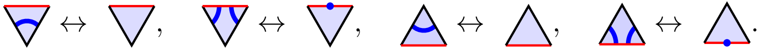
Elementary triangles in each configuration $C \in \mathcal{C}_m$ are replaced with:



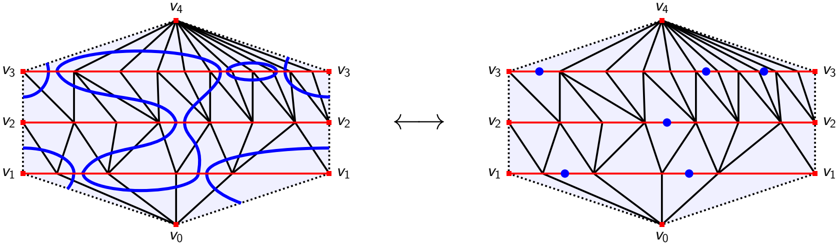
The resulting set of *dense loop model* configurations is denoted $\mathcal{L}_m^{de}$.



Dense loop configurations can be uniquely expressed by the *node* notation:



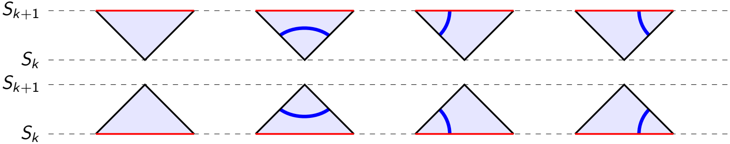
Returning to our previous example:



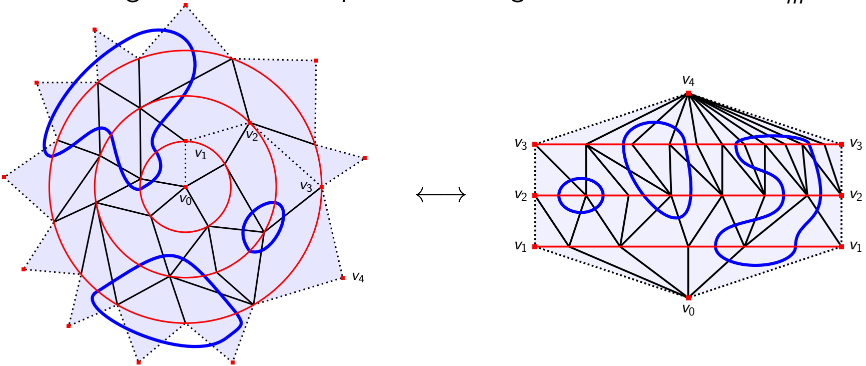
Each space-like edge can either be marked or unmarked.

Dilute loop model

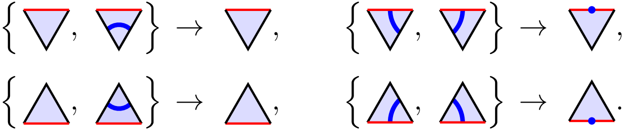
Elementary triangles in each configuration $C \in \mathcal{C}_m$ are replaced with:



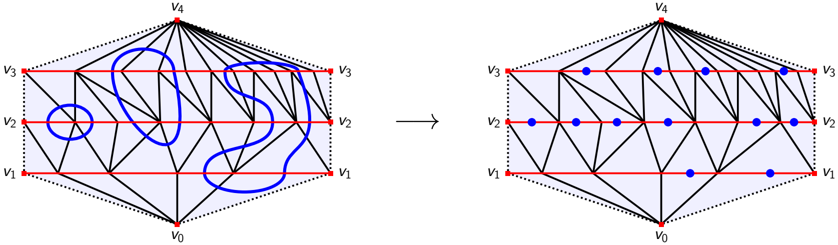
The resulting set of *dilute loop model* configurations is denoted $\mathcal{L}_m^{di}$.



Dilute loop configurations can be expressed by the node notation:

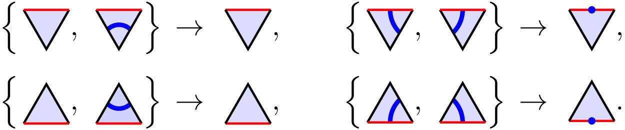


Returning to our previous example:

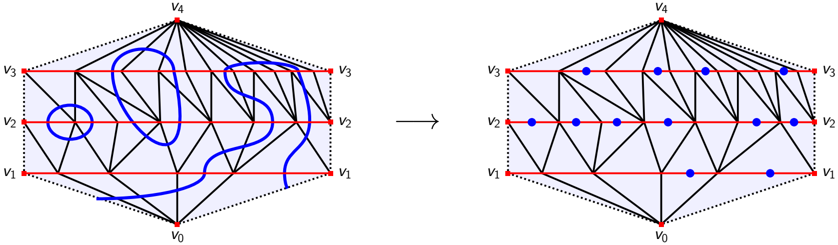


The loops are in 2^{m+1} to 1 correspondence with the nodes. There also exists a condition for nodes on each layer: $\#_k(\bullet) \in 2\mathbb{N}$, $1 \leq k \leq m$.

Dilute loop configurations can be expressed by the node notation:



Returning to our previous example:



The loops are in 2^{m+1} to 1 correspondence with the nodes. There also exists a condition for nodes on each layer: $\#_k(\bullet) \in 2\mathbb{N}$, $1 \leq k \leq m$.

Partition functions

For $C \in \mathcal{C}_m$, denote by $|C|$ the total number of space-like edges in C. The pure CDT partition functions are:

$$Z^p(g) := \sum_{m=0}^{\infty} Z_m^p(g)\,, \qquad Z_m^p(g) := \sum_{C \in \mathcal{C}_m} g^{|C|}.$$

For $L \in \mathcal{L}_m^\star$, denote by $\ell(L)$ the total number of loop-intersections in L. The loop model partition functions are:

$$Z^\star(g,\alpha) := \sum_{m=0}^{\infty} Z_m^\star(g,\alpha)\,, \quad Z_m^\star(g,\alpha) := \sum_{L \in \mathcal{L}_m^\star} g^{|L|}\alpha^{\ell(L)}.$$

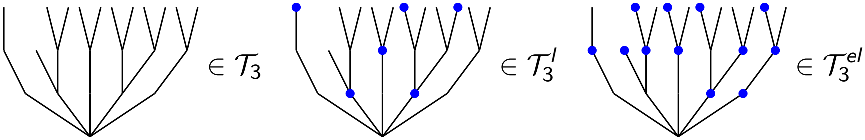
where $\star \in \{de, di\}$. Having defined partition functions relevant to each model. These functions need not be well defined for all $g, \alpha \in \mathbb{C}$.

$$Z^\bullet(g,\alpha) = \begin{cases} \text{Analytic,} & |g(\alpha)| < g_c(\alpha) \\ \text{Critical point/curve,} & g(\alpha) = g_c(\alpha) \\ \text{Divergent,} & g(\alpha) > g_c(\alpha) \end{cases}, \qquad \bullet \in \{p, de, di\}$$

Tree correspondences

Labelled planar trees

Let $\mathcal{T}_m$, $\mathcal{T}_m^l$ and $\mathcal{T}_m^{el}$ denote the set of height m unlabelled, (binary) labelled and even labelled planar trees respectively. For example:



For a tree T, denote by $V(T)$ and $\delta(T)$ the total number of vertices and labels respectively. To each set of trees $\mathcal{T}_m$, $\mathcal{T}_m^l$ and $\mathcal{T}_m^{el}$, we associate the partition functions

$$W(g) := \sum_{m=0}^{\infty} W_m(g), \qquad W_m(g) := \sum_{T \in \mathcal{T}_m} g^{V(T)},$$

$$W^\diamond(g,\alpha) := \sum_{m=0}^{\infty} W_m^\diamond(g,\alpha), \qquad W_m^\diamond(g,\alpha) := \sum_{T \in \mathcal{T}_m^\diamond} g^{V(T)} \alpha^{\delta(T)}$$

where $\diamond \in \{l, el\}$.

Tree correspondences

Theorem

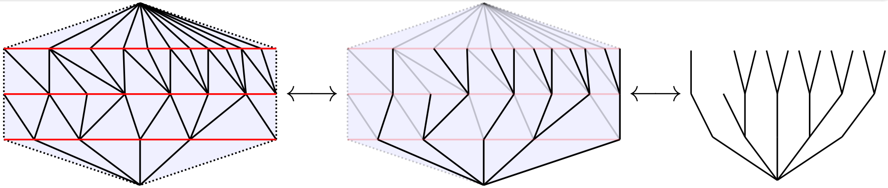
There exist correspondences between triangulations and trees:

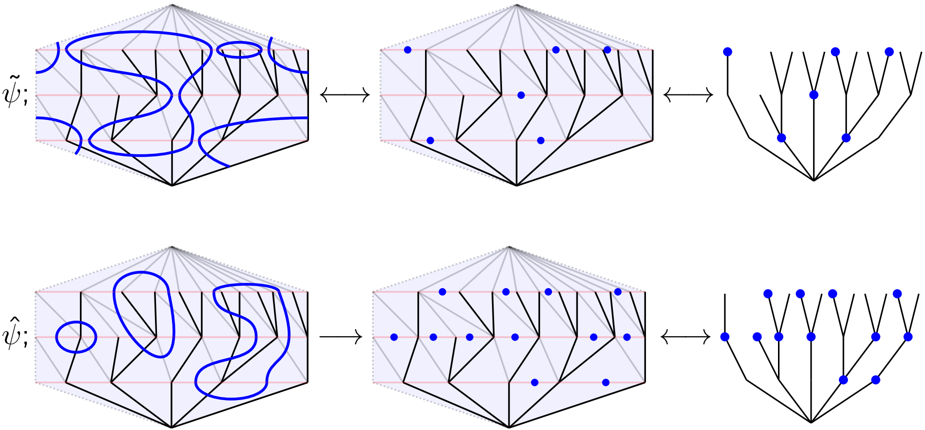
$$\psi : \mathcal{C}_m \to \mathcal{T}_m, \qquad \tilde{\psi} : \mathcal{L}_m^{de} \to \mathcal{T}_m^{l}, \qquad \hat{\psi} : \mathcal{L}_m^{di} \to \mathcal{T}_m^{el}$$

where ψ and $\tilde{\psi}$ are 1 to 1, while $\hat{\psi}$ is 2^{m+1} to 1. These are defined by:

- Remove all space-like (red) edges
- For each vertex, remove the leftmost outward-pointing time-like (black) edge
- Label each vertex to the right of an intersected space-like edge

ψ;





It follows that the partition functions are related by

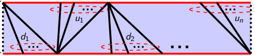
$$Z^p(g) = W(g), \quad Z^{de}(g,\alpha) = W^l(g,\alpha^2), \quad Z^{di}(g,\alpha) = \sum_{m=0}^{\infty} 2^{m+1} W^{el}_m(g,\alpha)$$

Transfer-matrix formalism

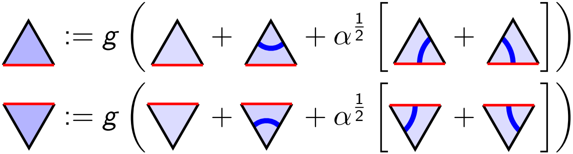
Transfer-matrix

The transfer-matrix is the combinatorial operator generating all infinitesimal configurations, the matrix elements are given by:

$$\mathsf{T}_{d,u} := \sum_{i,d_i,u_i} \mathrm{Coeff}\Big(\cdots \Big)$$

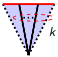


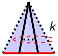
Coeff collects weights associated with the expansion:



Defining the vectors

$$\langle v| := \sum_{k\geq 1} \mathrm{Coeff}\Big(\cdots \Big), \qquad |v\rangle := \sum_{k\geq 1} \mathrm{Coeff}\Big(\cdots \Big),$$





the m height partition function of each model can be written

$$Z_m^{di}(g,\alpha) = \left\langle v \middle| \mathsf{T}^{m-1} \middle| v \right\rangle.$$

Critical behaviour

Pure CDT and dense loop model

Lemma

The planar tree partition functions admit the solution

$$W(g) = \frac{1-\sqrt{1-4g}}{2g}, \qquad W^{l}(g,\alpha) = \frac{1-\sqrt{1-4g(1+\alpha)}}{2g(1+\alpha)}$$

Recall the relations

$$Z^{p}(g) = W(g), \qquad Z^{de}(g,\alpha) = W^{l}(g,\alpha^2),$$

the pure CDT and dense loop model have the critical couplings $g_c = \frac{1}{4}$ and $g_c(\alpha) = \frac{1}{4(1+\alpha^2)}$, respectively. Expanding the partition functions

$$Z^{p}(g) = 2\sum_{n\geq 0}(-1)^n\left(\frac{g_c-g}{g_c}\right)^{\frac{n}{2}}, \quad Z^{de}(g,\alpha) = 2\sum_{n\geq 0}(-1)^n\left(\frac{g_c(\alpha)-g}{g_c(\alpha)}\right)^{\frac{n}{2}},$$

both have a critical exponent of $\frac{1}{2}$. With the Hausdorff dimension shown to be 2 almost surely, Durhuus, Jonsson and Wheater (2010).

Dilute loop model

The dilute model partition function cannot be related to the pure CDT model. Determining the critical behaviour we turn to the transfer-matrix.

$$Z^{di}(g,\alpha) = \sum_{m=0}^{\infty} 2^{m+1} W_m^{el}(g,\alpha) = 1 + \sum_{m=1}^{\infty} \left\langle v^{di} \middle| \mathsf{T}^{m-1} \middle| v^{di} \right\rangle.$$

The operator T admits the factorisation

$$\mathsf{T}(g,\alpha) = 2DK(g,\alpha),$$

Proposition

The operator $D^{\frac{1}{2}} K D^{\frac{1}{2}}$ is diagonalisable and analytic on the disk

$$\mathbb{D}_\alpha = \left\{ (g,\alpha) \in \mathbb{C}^2 \,|\, |g| < \tfrac{1}{4(1+|\alpha|)} \right\},$$

There exists a largest eigenvalue $\lambda_1(g,\alpha)$, it is an increasing function of g.

Corollary

For $C'(\alpha) > 0$, the dilute model partition function satisfies

$$Z^{di}(g,\alpha) \geq \frac{C'(\alpha)}{1-2\lambda_1} - B(g,\alpha),$$

where $B(g,\alpha)$ is bounded for g close to $g_c^{di}(\alpha)$.

With $\alpha \in [0,1]$ define

$$\overline{\lambda}_1(\alpha) := \lim_{g \nearrow \frac{1}{4(1+\alpha)}} \lambda_1(g,\alpha), \qquad \overline{\lambda}_1(0) = 1, \quad \overline{\lambda}_1(1) = \frac{1}{2}.$$

For α such that $\overline{\lambda}_1(\alpha) > \frac{1}{2}$ the critical coupling $g_c^{di}(\alpha)$ follows from

$$\lambda_1\left(g_c^{di}(\alpha),\alpha\right) = \frac{1}{2}, \qquad \text{where } g_c^{di}(\alpha) < \frac{1}{4(1+\alpha)}.$$

There exist two possibilities:

(i) $\overline{\lambda}_1(\alpha) = \frac{1}{2}$ for $\alpha = 1$, the constraint holds for all $\alpha \in [0,1)$.

(ii) $\overline{\lambda}_1(\alpha) = \frac{1}{2}$ for $\alpha \in [\alpha_0, 1]$ $0 < \alpha_0 < 1$, suggesting a phase transition.

Critical exponent

Establishing an upper bound we observe $Z_m^{di}(g,\alpha) \leq 2\big(\mathrm{tr}\mathsf{T}(g,\alpha)\big)^m$.

Theorem

For α real and sufficiently small, there exist $C_1(\alpha), C_2(\alpha) > 0$ such that

$$\frac{C_1(\alpha)}{g_c^{di}(\alpha) - g} \leq Z^{di}(g,\alpha) \leq \frac{C_2(\alpha)}{g_c^{di}(\alpha) - g}.$$

It follows that there exists an α such that the critical exponent of the dilute loop model is -1! Inducing a shift from $\frac{1}{2}$ of the pure CDT model.

Accompanying this shift is a change in Hausdorff dimension from 2 to 1, Durhuus and Ünel (2021).

Conclusion

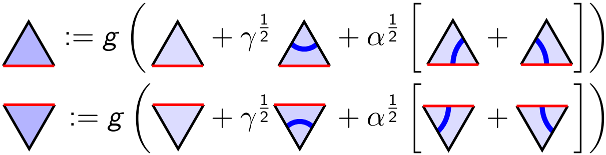
Conclusion

Summary:

- Pure CDT and dense loop models possess identical critical behaviour
- For α small, the critical behaviour of the dilute loop model is distinct from pure CDT suggesting a non-trivial loop-triangulation coupling
- This coupling induces a change in Hausdorff dimension from 2 to 1

Future direction:

- Examine $\overline{\lambda}_1(\alpha)$ further to investigate the presence of α_0
- Analyse a generalisation of the dilute model introducing a γ:



- Consider other loop models on triangulations incorporating a braid e.g the Brauer and BMW algebras

Thank you!